

THE CURVE CONE OF ALMOST COMPLEX 4-MANIFOLDS

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ABSTRACT. In this paper, we study the curve cone of an almost complex 4-manifold which is tamed by a symplectic form. In particular, we prove the cone theorem as in Mori theory for all such manifolds using the Seiberg-Witten theory. For small rational surfaces and minimal ruled surfaces, we study the configuration of negative curves. As an application, we prove the Nakai-Moishezon type duality for all almost Kähler structures on $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with $k \leq 9$ and minimal ruled surfaces with a negative curve. This is proved using a version of Gram-Schmidt orthogonalization process for the J -tamed symplectic inflation.

1. INTRODUCTION

The study of the curve cone is of much importance to the birational geometry of an algebraic variety. The cone theorem for smooth varieties proved in [25], which describes the structure of the curve cone by extremal rays, was the first major step of Mori's program. It was later generalized to a larger class of varieties by Kollár, Reid, Shokurov and others. The proof for a general variety relies on the bend-and-break technique, where a characteristic 0 proof is still lacking. However, there is an elementary proof for algebraic surfaces, see *e.g.* [27]. Early applications of the notion of the curve cone include Nakai-Moishezon's and Kleiman's ampleness criteria.

We could also similarly define the curve cone $A_J(M)$ for a tamed almost complex manifold (M, J) :

$$A_J(M) = \left\{ \sum a_i [C_i] \mid a_i > 0 \right\}$$

where C_i are irreducible J -holomorphic subvarieties on M . Here an irreducible J -holomorphic subvariety is the image of a J -holomorphic map $\phi : \Sigma \rightarrow M$ from a complex connected curve Σ , where ϕ is an embedding off a finite set. More generally, a J -holomorphic subvariety is a finite set of pairs $\{(C_i, m_i), 1 \leq i \leq n\}$, where each C_i is irreducible J -holomorphic subvariety and each m_i is a non-negative integer. Later, we sometimes say J -holomorphic curves (or simply, curves) instead of J -holomorphic subvarieties.

We will focus on dimension 4. Hence by taking Poincaré duality (we will identify the curve classes with their Poincaré dual cohomology classes by abusing the notation), we have $A_J(M)$ sitting as a cone in vector space $H_J^+(M) \subset H^2(M; \mathbb{R})$. Here $H_J^+(M)$ is called the J -invariant cohomology

which is introduced in [18, 8] along with the J -anti-invariant $H_J^-(M)$. The almost complex structure J acts on the bundle of real 2-forms Λ^2 as an involution, by $\alpha(\cdot, \cdot) \rightarrow \alpha(J\cdot, J\cdot)$. This involution induces the splitting into J -invariant, respectively, J -anti-invariant 2-forms $\Lambda^2 = \Lambda_J^+ \oplus \Lambda_J^-$. Then we define

$$H_J^\pm(M) = \{\mathfrak{a} \in H^2(M; \mathbb{R}) \mid \exists \alpha \in \Lambda_J^\pm, d\alpha = 0 \text{ such that } [\alpha] = \mathfrak{a}\}.$$

Usually, to prove a geometric result for a general (non-generic) tamed almost complex structure is far more delicate than the generic case, since we have to deal with very general subvarieties. The techniques developed in [19, 20] enable us to have a fairly clear structural picture for subvarieties in a J -nef class. In this paper, we would develop more techniques to work with a general tamed almost complex structure. In particular, we find that the curve cone and sometimes curve configurations behave in many aspects similar to that of algebraic varieties.

Throughout the paper, all the numerical calculations have the following geometric picture in mind: we think an element in $A_J(M)$ (or $A_J^\mathbb{Q}(M) = \overline{A}_J(M) \cap H^2(M, \mathbb{Q})$) as the homology class of an \mathbb{R} (or \mathbb{Q}) J -holomorphic subvarieties. Here an \mathbb{R} (or \mathbb{Q}) subvariety allows m_i in the definition of subvariety to be positive real numbers (or rational numbers). With this freedom in hand, we can choose some or all “irreducible components” C_i to be the extremal rays of $A_J(M)$. Here a subcone $N \subset A_J(M)$ is called *extremal* if $u, v \in A_J(M), u + v \in N$ imply that $u, v \in N$. A 1-dimensional extremal subcone is called a *extremal ray*. Hence, the numerical information of the given class in $A_J(M)$ would give information for these extremal rays, and vice versa. The geometry is thus hidden in the numerical information. A typical example is Corollary 2.10 which guarantees the existence of -1 rational curves by the numerical property of a class known to be in $A_J(M)$.

From another viewpoint, considering \mathbb{Q} -subvarieties of an integral class $e \in A_J(M)$ is the same as considering subvarieties in all the integral classes of the ray $\mathbb{R}^+ \cdot e \subset A_J(M)$. The complexity arises, thus one has to study all integral classes in the ray rather than just a single class, because the subvarieties in a class and its multiples do not change linearly, see *e.g.* Examples 2.6, 2.11.

For this sake, we will first understand the extremal rays. Our first result, the cone theorem for tamed almost complex 4-manifolds, is a general structural result on the extremal rays of the “negative” part (this is not to be confused with the notion of “negative curves” used later, which means curves with negative self-intersection). There are several different proofs of the cone theorem in algebraic geometry (at least for algebraic surfaces). Mori’s original proof uses the bend-and-break and the Kleiman’s ampleness criterion. The argument in [27] is bend-and-break free, but it proves rationality theorem first. Our proof proceeds in a totally different logical order. We do not need all these technical results before the cone theorem. In our paper, the rationality theorem, Proposition 2.9, is proved as a corollary of

the cone theorem. Moreover, we are only able to prove Nakai-Moishezon and Kleiman type results for certain rational surfaces.

However, the soul of the algebraic geometric proofs and our proof is the same: vanishing \Rightarrow non-vanishing \Rightarrow cone theorem. We start by proving certain Seiberg-Witten invariants vanish. Using the Seiberg-Witten wall crossing formula [14], we have non-vanishing results. Then Taubes' SW=Gr [28, 29] would imply a class does not span an extremal ray if it could be written as the sum of two classes with non-trivial Seiberg-Witten invariants. This would eventually guarantee the extremal rays of the “negative” part are generated by rational curves. The precise statement of the cone theorem is the following, which is a combination of Theorem 2.4 and Proposition 2.7.

Theorem 1.1. *Let (M, J) be a tamed almost complex 4-manifold. Then*

$$\overline{A}_J(M) = \overline{A}_{K_J}(M) + \sum \mathbb{R}^+[L_i]$$

where $L_i \subset M$ are countably many smooth irreducible rational curves such that $-3 \leq K_J \cdot [L_i] < 0$ which span the extremal rays $\mathbb{R}^+[L_i]$ of $\overline{A}_J(M)$.

Moreover, for any J -almost Kähler symplectic form ω and any given $\epsilon > 0$, there are only finitely many extremal rays with $(K_J + \epsilon[\omega]) \cdot [L_i] \leq 0$.

In addition, an irreducible curve C is an extremal rational curve if and only if

- (1) C is a -1 rational curve;
- (2) M is a minimal ruled surface or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, and C is a fiber;
- (3) $M = \mathbb{C}P^2$ and C is a projective line.

In fact, we prove a slightly stronger version of cone theorem:

$$\overline{A}_J^{\mathbb{Q}}(M) = \overline{A}_{K_J}^{\mathbb{Q}}(M) + \sum \mathbb{Q}^+[L_i],$$

where $\overline{A}_J^{\mathbb{Q}}(M) = \overline{A}_J(M) \cap H^2(M, \mathbb{Q})$ and $\overline{A}_{K_J}^{\mathbb{Q}}(M) = \overline{A}_{K_J}(M) \cap H^2(M, \mathbb{Q})$.

Here K_J is the canonical class of (M, J) and $A_{K_J}(M) = \{C \in A_J(M) | K_J \cdot C \geq 0\}$ is the “positive” part of the curve cone. In general, this part is not generated by countably many extremal rays since it may have round boundary. This phenomenon happens in particular when we do not have sufficient curves as for a generic almost complex structure on manifolds with $b^+ > 1$, or even in the rational surfaces $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with $k > 9$.

On the other hand, there are indeed many cases whose curve cones are polytopes. The most well known examples are rational surfaces $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ when $k \leq 9$ and $S^2 \times S^2$. The cases of S^2 bundles over S^2 are treated in [19] for example.

We give a careful analysis of the curve cone for all possible tamed almost complex structures on $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ in Section 3. In particular, we show that there are at least two embedded -1 rational curves for any tamed almost complex structures (Theorem 3.6). This should be compared to other results on the existence of embedded -1 rational curves. As a corollary of Theorem 1.1, we give a proof of the fact that any tamed J on a non-minimal symplectic

4-manifold which is not diffeomorphic to a S^2 bundle contains at least one smooth -1 rational curve. In fact, it appears as a component of a \mathbb{Q} -variety representing an exceptional sphere class (Corollary 2.10). On the other hand, based on an argument communicated to the author by Dusa McDuff, we show that if M is not diffeomorphic to one point blow up of S^2 bundles, then there exists a tamed J such that all -1 rational curves are disjoint and any possible numbers of such -1 rational curves could be realized (Theorem 4.13).

Let $h_J^+(M) = \dim H_J^+(M)$. Then by the light cone lemma, when $h_J^+(M) = b^-(M)+1$ and $b^-(M) > 1$, the boundary hyperplanes of the dual of the curve cone are determined by J -holomorphic curves with negative self-intersection, *i.e.* negative curves (see Lemma 5.1). The equality $h_J^+(M) = b^-(M) + 1$ holds for any complex structure J or manifolds with $b^+(M) = 1$. Hence to determine various cones, *e.g.* the curve cone and the almost Kähler cone, for $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ when $k \leq 9$, we are reduced to determine all possible negative curves. The classification is done in Section 4. Especially, all negative curves (resp. non-positive curves) on $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with $k \leq 9$ (resp. $k < 9$) are rational curves. A complete list of the classes of all such rational curves are obtained in Propositions 4.1, 4.4, 4.5. Moreover, we have the following more precise statement about negative curve configurations.

Theorem 1.2. *For rational 4-manifolds $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with $k < 8$, the set of all the possible configurations of negative self-intersection curves for tamed almost complex structures are the same as the set for complex structures.*

Namely, given a tamed almost complex structures, we could find a complex structure (on the same manifold) such that their configurations of negative curves are the same. Here by a configuration of negative curves, we mean the set of homology classes and multiplicities of each intersection point.

This is also true for minimal ruled surfaces. However, as observed in [5], this is not true for a non-minimal ruled surface, *e.g.* $(T^2 \times S^2) \# \overline{\mathbb{C}P^2}$. A generic almost complex structure will have only two negative curves E and $F - E$ where F is class of S^2 and E is the class of exceptional curve, while there is no complex structure on it having exactly those two negative curves. The paper [5] also gives such an example which is a minimal surface of general type. Currently, no such simply connected examples are known. On $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with $k \geq 9$, it is related to the Nagata conjecture.

Question 1.3. *(Nagata) For every $k > 9$, do we have complex structure on $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$, every irreducible curve C , such that*

$$d \geq \frac{\sum_{q=1}^k m_q}{\sqrt{k}},$$

where $[C] = dH - \sum_{q=1}^k m_q E_q$?

This is equivalent to say that $H - \frac{1}{\sqrt{k}} \sum E_i$ is on the closure of the Kähler cone by Nakai-Moishezon criterion. It is easy to see that a generic tamed almost complex structure satisfies the inequality.

The above discussion gives us a clearer picture on the polytopic boundary of the curve cone $A_J(M)$. We can apply it to understand the Nakai-Moishezon or the Kleiman type duality between the curve cone and the almost Kähler cone $\mathcal{K}_J^c = \{[\omega] \in H^2(M; \mathbb{R}) | \omega \text{ is compatible with } J\}$. When $b^+(M) = 1$, it is shown in [18] that \mathcal{K}_J^c is equal to the tame cone $\mathcal{K}_J^t = \{[\omega] \in H^2(M; \mathbb{R}) | \omega \text{ tames } J\}$.

Let us introduce a couple more cones. First is the positive cone $\mathcal{P} = \{e \in H^2(M; \mathbb{R}) | e \cdot e > 0\}$. The second cone $A_J^{\vee, > 0}(M)$ (resp. $\overline{A}_J^{\vee, > 0}(M)$) is the positive dual of $A_J(M)$ (resp. $\overline{A}_J(M)$) where the duality is taken within $H_J^+(M)$. Let $\mathcal{P}_J = A_J^{\vee, > 0}(M) \cap \mathcal{P}$. Clearly, $\mathcal{K}_J^c \subset \mathcal{P}_J$. Then we can ask the following

Question 1.4. *For an almost Kähler structure J on a closed, oriented 4-manifold M with $b^+(M) = 1$, is*

$$\mathcal{K}_J^c = \mathcal{P}_J = \overline{A}_J^{\vee, > 0}(M)?$$

When $b^+(M) > 1$, is \mathcal{K}_J^c a connected component of \mathcal{P}_J ?

When J is an integrable complex structure, \mathcal{K}_J^c is the Kähler cone. A Kählerian Nakai-Moishezon theorem is given by Buchdahl and Lamari in dimension 4 [3, 12], and Demailly-Paun [6] in arbitrary dimension to determine the Kähler cone completely.

It is worth noting that in algebraic geometry, all the proofs of the cone theorem relies on the Nakai-Moishezon or Kleiman ampleness criterion. However, our almost complex cone theorem (Theorem 1.1) does not need the corresponding version, *i.e.* Question 1.4.

The key of Question 1.4 is to construct almost Kähler forms. There are currently two methods of construction. The first is using Taubes' subvarieties-current-form technique [30]. However, to argue it for an arbitrary J , we have to use spherical subvarieties as in [19]. It was successfully used to affirm Question 1.4 for S^2 bundles over S^2 . The method is applied to prove it for $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ in this paper. However, it cannot go further along this line. The limit of this method is discussed in Section 4.2.

The second method is reducing the construction of the almost Kähler cone to determine the more flexible tame cone \mathcal{K}_J^t by virtue of the identity $\mathcal{K}_J^c = \mathcal{K}_J^t \cap H_J^+(M)$ established in [18]. In this situation, we could apply the J -tamed symplectic inflation developed by McDuff [22] and Buse [4]. More precisely, we apply Buse's inflation for negative J -holomorphic curves to our extremal ray of curve cone. The combinatorial version of it, the formal J -inflation, is introduced. A version of Gram-Schmidt orthogonalization process is applied in our situation to significantly simplify the otherwise too complicated calculation. We are able to prove the following general result when the curve cone has no round boundary.

Theorem 1.5. *On a 4-dimensional almost complex manifold (M, J) with $\mathcal{K}_J^c \neq \emptyset$, and $h_J^+ = b^- + 1$, if \mathcal{P}_J has no round boundary and each of the boundary hyperplane is determined by a smooth negative curve, then \mathcal{K}_J^c is a connected component of \mathcal{P}_J .*

Applying the discussion of the curve cone for rational and ruled surfaces in Sections 2-4, we have the both Nakai-Moishezon and Kleiman dualities.

Theorem 1.6. *If J is almost Kähler on $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with $k \leq 9$ or a minimal irrational ruled surface with a negative curve, then*

$$\mathcal{K}_J^c = \mathcal{K}_J^t = \mathcal{P}_J = \overline{A}_J^{\vee, >0}(M).$$

We are also able to show that Question 1.4 is true for a generic tamed J on a manifold with $b^+(M) = 1$. Here a generic tamed almost complex structure means that it is chosen from a residual subset of all tamed almost complex structures.

Theorem 1.7. *On a symplectic 4-manifold with $b^+ = 1$, $\mathcal{K}_J^t = \mathcal{K}_J^c = \mathcal{P}_J$ for a generic tamed J .*

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2. THE CONE THEOREM

In this section, we will give a proof of the cone theorem for tamed almost complex 4-manifolds. Recall an almost complex structure J is said to be tamed if there is a symplectic form ω such that the bilinear form $\omega(\cdot, J(\cdot))$ is positive definite. A taming form of J is said to be compatible with J if the bilinear form $\omega(\cdot, J(\cdot))$ is symmetric. If there is a symplectic form compatible with the almost complex structure J , then J is called almost Kähler. Hence, a J -compatible symplectic form is also called a J -almost Kähler form.

Given a class $e \in H_2(M; \mathbb{Z})$, introduce the J -genus of e ,

$$g_J(e) = \frac{1}{2}(e \cdot e + K \cdot e) + 1,$$

where $K = K_J$ is the canonical class of J . Moreover, when C is an irreducible subvariety, $g_J([C])$ is non-negative. In fact, if Σ is the model curve of C , by the adjunction inequality,

$$(1) \quad g_J(e_C) \geq g(\Sigma),$$

with equality if and only if C is smooth. In particular, if C is the class of an irreducible curve having $C^2 < 0$ and $K \cdot C < 0$ then the curve must be a -1 rational curve.

In the following, a rational curve means an irreducible J -holomorphic subvariety of J -genus 0. By (1), such a curve has to be smooth.

2.1. The Seiberg-Witten invariant. In this subsection, we will give a very brief introduction to the Seiberg-Witten invariant, which will be the main tool to establish Theorem 1.1. For a detailed introduction, see for example [14, 15] and references therein.

Let M be an oriented 4-manifold with a given Riemannian metric g and a spin^c structure \mathcal{L} on M . Hence there are a pair of rank 2 complex vector bundles S^\pm with isomorphisms $\det(S^+) = \det(S^-) = \mathcal{L}$. The Seiberg-Witten equations are for a pair (A, ϕ) where A is a connection of \mathcal{L} and ϕ is a section of S^+ . These equations are

$$D_A \phi = 0$$

$$F_A^+ = iq(\phi) + i\eta$$

where q is a canonical map $q : \Gamma(S^+) \rightarrow \Omega_+^2(M)$ and η is a self-dual 2-form on M .

The group $C^\infty(M; S^1)$ naturally acts on the space of solutions and acts freely at irreducible solutions. Recall a reducible solution has $\phi = 0$, and hence $F_A^+ = i\eta$. The quotient is the moduli space and is denoted by $\mathcal{M}_M(\mathcal{L}, g, \eta)$. For generic pairs (g, η) , the Seiberg-Witten moduli space $\mathcal{M}_M(\mathcal{L}, g, \eta)$ is a compact manifold of dimension

$$2d(\mathcal{L}) = \frac{1}{4}(c_1(\mathcal{L})^2 - (3\sigma(M) + 2\chi(M)))$$

where $\sigma(M)$ is the signature and $\chi(M)$ is the Euler number. Furthermore, an orientation is given to $\mathcal{M}_M(\mathcal{L}, g, \eta)$ by fixing a homology orientation for M , *i.e.* an orientation of $H^1(M) \oplus H_+^2(M)$. The space of g -self-dual space $\mathcal{H}_g^+(M)$ is spanned by a single harmonic 2-form ω_g of norm 1 agreeing with the homology orientation.

Now, for the convenience of notation, we fix a (homotopy class of) almost complex structures. In particular, it determines a canonical class K which is the first Chern class of the cotangent bundle. We denote $e := \frac{c_1(\mathcal{L}) + K}{2}$. In the following, notice the first Chern class $c_1(\mathcal{L})$ determines \mathcal{L} and *vice versa*. For a generic choice of (g, η) , the Seiberg-Witten invariant $SW_{M,g,\eta}(e)$ is defined as follows. If $d(\mathcal{L}) < 0$, then the SW invariant is zero. If $d(\mathcal{L}) = 0$, then the moduli space is a finite union of signed points and the SW invariant is the sum of the corresponding signs. If $d(\mathcal{L}) > 0$, then the SW invariant is obtained by pairing the fundamental class of $\mathcal{M}_M(\mathcal{L}, g, \eta)$ with the maximal cup product of the Euler class of the S^1 -bundle $\mathcal{M}_M^0(\mathcal{L}, g, \eta)$ over $\mathcal{M}_M(\mathcal{L}, g, \eta)$. Here $\mathcal{M}_M^0(\mathcal{L}, g, \eta)$ is called the based moduli space which is the quotient of the space of solution by $C_0^\infty(M, S^1)$ (the elements in $C^\infty(M, S^1)$ which map a base point in M to 1 in S^1).

If $b^+ > 1$, a generic path of (g, η) contains no reducible solutions. Hence, the Seiberg-Witten invariant is an oriented diffeomorphism invariant in this

case. Hence we can use the notation $SW(e)$ for the Seiberg-Witten invariant. We will also write

$$\dim_{SW}(e) = 2d(\mathcal{L}) = \frac{1}{4}((2e - K)^2 - K^2) = e^2 - K \cdot e$$

for the Seiberg-Witten dimension. In the case $b^+ = 1$, there might be reducible solutions on a 1-dimensional family. Recall that the curvature F_A represents the cohomology class $-2\pi i c_1(\mathcal{L})$. Hence $F_A^+ = i\eta$ holds only if $-2\pi c_1(\mathcal{L})^+ = \eta$. This happens if and only if the discriminant $\Delta_{\mathcal{L}}(g, \eta) := \int (2\pi c_1(\mathcal{L}) + \eta)\omega_g = 0$. With this in mind, the set of pairs (g, η) with positive (resp. negative) discriminant is called the positive (resp. negative) \mathcal{L} chamber. We use the notation $SW^{\pm}(e)$ for the Seiberg-Witten invariants in these two chambers. Moreover, in the this paper, we will use $SW(e)$ instead of $SW^-(e)$ when $b^+ = 1$.

As one can easily see from the discussion above, SW (or SW^{\pm}) could be defined more generally as map from $H^2(M, \mathbb{Z})$ to $\Lambda^* H^1(M, \mathbb{Z})$, although we do not need this generality for this paper.

We now assume (M, ω) is a symplectic 4-manifold, and J is a ω -tamed almost complex structure. Then the theorems in [29] and [15] equate Seiberg-Witten invariants with Gromov-Taubes invariants that are defined by making a suitably counting of J -holomorphic subvarieties (In fact, the $SW(e)$ we defined here is essentially the Gromov-Taubes invariant in the literature). Especially, when $SW(e) \neq 0$, there is a J -holomorphic subvariety in class e passing through $\dim_{SW}(e)$ given points. This is the key result from the Seiberg-Witten theory we will use in this section.

2.2. The cone theorem. We first have the following

Lemma 2.1. *Let J be a tamed almost complex structure on a symplectic 4-manifold which is not rational or ruled. Let C be an irreducible J -holomorphic curve with $K \cdot [C] < 0$. Then C has to be a -1 rational curve.*

Proof. If M is a minimal symplectic 4-manifold which is not rational or ruled, then by Taubes' theorem [28] we always have a J -holomorphic subvariety in class K when $b^+ > 1$ and by Li-Liu's theorem in class $2K$ when $b^+ = 1$. If $2K = \sum a_i [C_i]$, then $K \cdot C < 0$ would imply C is one of C_i . Let it be C_1 . Then $[C] \cdot (2K - a_1 [C]) \geq 0$ implies $C^2 < 0$. Altogether, we have C is a -1 rational curve. This is a contradiction.

If M is not minimal, then K could be written as $K_0 + E_1 + \cdots + E_k$, where $K_0 \cdot E_i = 0$ and E_i are the exceptional classes. We also have $2K_0 = \sum a_i [C_i]$. Thus $2K$ is also a linear combination of curve classes. The same argument as in the minimal case implies C is a -1 rational curve. \square

Next we prove a very useful lemma. In many applications later, our C also has $SW([C]) \neq 0$.

Lemma 2.2. *If C is an irreducible J -holomorphic curve with $C^2 \geq 0$ and $SW(e) \neq 0$, then $e \cdot [C] \geq 0$.*

Proof. Since $SW(e) \neq 0$, we can represent e by a possible reducible J -holomorphic subvariety. Since each irreducible curve C' has $[C'] \cdot [C] \geq 0$, we have $e \cdot [C] \geq 0$. \square

Proposition 2.3. *Let (M, J) be a tamed almost complex 4-manifold. Let C be an irreducible curve such that $K \cdot [C] < 0$. Then $SW([C]) \neq 0$. Moreover, there is a curve in class $[C]$ passing through any given point when C is not a -1 rational curve.*

Proof. First we could assume M is rational or ruled. Otherwise, it is proved in Lemma 2.1.

$$\text{Notice } C^2 + K \cdot [C] = 2g - 2,$$

$$[C] \cdot (K - [C]) = -\dim_{SW}[C] = -\dim_{SW}(K - [C]) = 2K \cdot [C] + 2 - 2g \leq 0.$$

The equality holds if and only if $g = 0$, $K \cdot [C] = -1$, and $C^2 = -1$, i.e. C is a -1 rational curve. Otherwise $C^2 \geq 2g - 2 + 2 \geq 0$.

Hence we assume $[C] \cdot (K - [C]) < 0$. If $SW(K - [C]) \neq 0$, by $SW=Gr$ we have a (possibly reducible) J -holomorphic curve in class $K - [C]$. Hence the irreducible curve C must be a component of this curve and $C^2 < 0$. This contradicts the adjunction and dimension formula. Thus $SW(K - [C]) = 0$.

Since $\dim_{SW}[C] \geq 0$, we have wall-crossing formula

$$|SW(K - [C]) - SW([C])| = \begin{cases} 1 & \text{if } (M, \omega) \text{ rational,} \\ |1 + [C] \cdot T|^h & \text{if } (M, \omega) \text{ rationally ruled,} \end{cases}$$

where T is the unique positive fiber class and h is the genus of base surface of rationally ruled manifolds (see [14]). In summary, $SW([C]) \neq 0$. Hence there is a curve in class $[C]$ passing through any given point since $\dim_{SW}[C] > 0$. \square

Next we will prove the first two statements of Theorem 1.1, the cone theorem.

Theorem 2.4. *Let (M, J) be a tamed almost complex 4-manifold. Then*

$$\overline{A}_J(M) = \overline{A}_{K_J}(M) + \sum \mathbb{R}^+[L_i]$$

where $L_i \subset M$ are countably many smooth irreducible rational curves such that $-3 \leq K \cdot [L_i] < 0$ which span the extremal rays $\mathbb{R}^+[L_i]$ of $\overline{A}_J(M)$.

Moreover, for any J -almost Kähler symplectic form ω and any given $\epsilon > 0$, there are only finitely many extremal rays with $(K + \epsilon[\omega]) \cdot [L_i] \leq 0$.

Proof. We first prove our statement of extremal rays $\mathbb{R}^+[L_i]$. When M is not rational or ruled, then Lemma 2.1 verifies our claim. Especially, L_i are finitely many rational curves with $K \cdot [L_i] = -1$.

In general, let C be an irreducible curve with $K \cdot [C] < 0$. By Proposition 2.3, we have $SW([C]) \neq 0$. Hence for any tamed almost complex structure J , there is a (possibly reducible) subvariety in class $[C]$ by Taubes' $SW=Gr$ and Gromov compactness. Especially, it is true for a projective variety.

Now, assume C is an irreducible curve with $g_J([C]) > 0$ and $K \cdot [C] < 0$, or $g_J([C]) = 0$ and $K \cdot [C] < -3$ on a rational or ruled surface. We want to show that $[C]$ cannot span an extremal ray of the curve cone (we will say $[C]$ is not extremal for simplicity). We divide our discussion into the following cases.

Case 1: Irrational ruled surfaces

- $M = \Sigma_h \times S^2$, $h > 1$, and its blowups

In this case, let U be the class of the base Σ_h and T be the class of the fiber S^2 . Then the canonical class $K = -2U + (2h - 2)T + \sum_i E_i$. Let $[C] = aU + bT - \sum_i c_i E_i$.

Since both $[C]$ and T pair negatively with K , by Lemma 2.3, both classes have non-trivial Seiberg-Witten invariant. Applying Lemma 2.2 to the pair $[C]$ and T , we have $a \geq 0$. We could assume $[C]$ is not one of the classes of exceptional curves E_i . Then applying Lemma 2.2 to $[C]$ and E_i gives $c_i \geq 0$.

The assumption $K \cdot [C] < 0$ reads as

$$a(2h - 2) - 2b + \sum c_i < 0.$$

Especially, we have $b > 0$.

When $a = 0$, we have

$$-\sum c_i^2 = C^2 = 2g(C) - 2 - K \cdot [C] > 2g(C) - 2 \geq -2.$$

It works only when $g(C) = 0$ and there is a unique nonzero c_i , say c_1 , which equals to 1. Moreover, $K \cdot [C] = -1$ would imply $[C] = T - E_1$.

When $a > 0$, we take the smooth projection $f : C \rightarrow \Sigma_h$ to the base. It has degree $a = [C] \cdot T$. Since Σ_h has genus greater than one, and by Kneser's theorem, we have

$$2g(C) - 2 \geq a(2h - 2) \geq 2a.$$

Now, we are planning to show that the class $[C] - T$ has non trivial Seiberg-Witten invariant. First we show $SW(K - ([C] - T)) = 0$. If not, notice $SW(T) \neq 0$, we should have $(K - ([C] - T)) \cdot T \geq 0$ by Lemma 2.2. However, it contradicts to the calculation $(K - ([C] - T)) \cdot T = -2 - [C] \cdot T < 0$. Next we show the Seiberg-Witten dimension of it is nonnegative.

$$\begin{aligned} \dim_{SW}([C] - T) &= ([C] - T)^2 - K \cdot ([C] - T) \\ &= C^2 - K \cdot [C] - 2 - 2[C] \cdot T \\ &\geq 2g(C) - 2 - 2a \\ &\geq 0. \end{aligned}$$

Finally the wall crossing formula implies

$$|SW([C] - T)| = |SW(K - ([C] - T)) - SW([C] - T)| = |1 + a|^h \neq 0.$$

Hence, we complete our argument that $[C] = ([C] - T) + T$ is not extremal in this case.

- $M = T^2 \times S^2$ and its blowups

We use the same setting as the above case. That is, we assume $[C] = aU + bT - \sum_i c_i E_i$ and the canonical class $K = -2U + \sum_i E_i$. As showed in the above case, we only need to show that when $a > 0$, $[C]$ is not extremal. Without loss, we assume $c_1 \geq c_2 \geq \dots$.

Notice

$$C^2 = 2ab - \sum c_i^2, \quad -K \cdot [C] = 2b - \sum c_i > 0.$$

We will first show that $[C]$ must not be extremal in the case that $c_1 \leq a$ and make a corresponding change in the $c_1 > a$ case later.

$$C^2 = 2ab - \sum c_i^2 \geq 2ab - a \sum c_i = a(2b - \sum c_i) = a(-K \cdot [C]).$$

Hence look at the classes $l[C] - T$, we have

$$\begin{aligned} \dim_{SW}(l[C] - T) &= (l[C] - T)^2 - K \cdot (l[C] - T) \\ &= (l^2 C^2 - lK \cdot [C]) - 2 - 2l[C] \cdot T \\ &\geq (l^2 a + l)(-K \cdot [C]) - 2 - 2la \\ &\geq l^2 a + l - 2la - 2 \end{aligned}$$

It is greater than 0 if l is large enough (e.g. $l > 2a$).

Since $(K - (l[C] - T)) \cdot T = -2 - la < 0$, Lemma 2.2 implies $SW(K - (l[C] - T)) = 0$. Apply the wall crossing formula

$$|SW(l[C] - T)| = |SW(K - (l[C] - T)) - SW(l[C] - T)| = |1 + la| \neq 0.$$

Hence $[C] = \frac{1}{l}((l[C] - T) + T)$ where $l[C] - T$ is not proportional to T since $a > 0$. It is a decomposition of two non-proportional classes with non-trivial Seiberg-Witten invariant which is not extremal.

If $c_1 > a$. similar to the above, we first calculate the Seiberg-Witten dimension of the class $[C] - T + E_1$.

$$\begin{aligned} \dim_{SW}([C] - T + E_1) &= ([C] - T + E_1)^2 - K \cdot ([C] - T + E_1) \\ &= (C^2 - K \cdot [C]) - 2 - 2[C] \cdot T + 2c_1 \\ &\geq C^2 - K \cdot [C] \\ &> 0. \end{aligned}$$

Again $(K - ([C] - T + E_1)) \cdot T = -2 - a < 0$ implies $SW(K - ([C] - T + E_1)) = 0$. Then

$$|SW([C] - T + E_1)| = |SW(K - ([C] - T + E_1)) - SW(l[C] - T + E_1)| = |1 + a| \neq 0.$$

And $[C] = ([C] - T + E_1) + (T - E_1)$ is not extremal.

- M is a non-trivial S^2 bundle over Σ_h , $h \geq 1$

Since the blow-ups of it are diffeomorphic to those of trivial bundle, we are only left with the non-trivial bundles.

Let U be the class of a section with $U^2 = 1$ and T be the class of the fiber. Then $K = -2U + (2h - 1)T$. Let $[C] = aU + bT$. Again, we have $a \geq 0$. The condition $K \cdot [C] < 0$ reads as

$$2b > a(2h - 3).$$

If $a = 0$,

$$0 = C^2 = 2g(C) - 2 - K \cdot [C] > 2g(C) - 2.$$

Thus $g(C) = 0$ and $K \cdot C = -2$.

If $a > 0$ and $h > 1$, then we have $b > 0$. Hence $C^2 = a^2 + 2ab \geq 1 + 2a = 1 + 2[C] \cdot T$. Hence $\dim_{SW}([C] - T) = C^2 - K \cdot [C] - 2 - 2[C] \cdot T \geq 0$. Moreover $SW(K - ([C] - T)) = 0$, since $(K - ([C] - T)) \cdot T = -2 - a < 0$. Then the wall crossing formula implies

$$|SW([C] - T)| = |SW(K - ([C] - T)) - SW([C] - T)| = |1 + a|^h \neq 0.$$

And $[C] = ([C] - T) + T$ is not an extremal ray.

If $a > 0$ and $h = 1$, then we have

$$C^2 = a^2 + 2ab > 0, \quad -K \cdot [C] = 2b + a > 0.$$

Look at the classes $l[C] - T$, we have

$$\begin{aligned} \dim_{SW}(l[C] - T) &= (l[C] - T)^2 - K \cdot (l[C] - T) \\ &= (l^2 C^2 - lK \cdot [C]) - 2 - 2l[C] \cdot T \\ &= (l^2 a + l)(a + 2b) - 2 - 2la \\ &\geq l^2 a + l - 2la - 2 \end{aligned}$$

It is greater than 0 if $l > 2a$ is large enough. Moreover $SW(K - (l[C] - T)) = 0$, since $(K - (l[C] - T)) \cdot T = -2 - la < 0$. Then the wall crossing formula shows

$$|SW(l[C] - T)| = |SW(K - (l[C] - T)) - SW(l[C] - T)| = |1 + la|^h \neq 0.$$

Hence $[C] = \frac{1}{l}((l[C] - T) + T)$ is not extremal.

Note our argument shows more specifically that any extremal ray paring negatively with the canonical class K must be generated by either E_i , $T - E_i$ or in the minimal case T .

Case 2: Rational surfaces:

Recall C is an irreducible curve with $K \cdot [C] < 0$, which would imply $SW([C]) \neq 0$ by Lemma 2.3. Suppose C is not a rational curve with self-intersection 0 or -1 . Since $SW([C]) \neq 0$ and $K \cdot [C] < 0$, by Lemma 2.2, we know $[C]$ is in the closure of the cone

$$P_{K_J} := \{e \in H^2(M; \mathbb{R}) | e^2 > 0, e \cdot E > 0 \text{ for any } E \in \mathcal{E}_{K_J}, e \cdot (-K_J) > 0\}$$

where \mathcal{E}_{K_J} is the set of -1 symplectic rational curves. Let S be the set of homology classes which are represented by smoothly embedded spheres. We define

$$S_{K_J}^+ = \{e \in S | g_J(e) = 0, e^2 > 0\}.$$

Using this notation,

$$\mathcal{E}_{K_J} = \{e \in S | g_J(e) = 0, e^2 = -1\}.$$

By Proposition 5.20 in [19], $P_{K_J} = S_{K_J}^+$ where the latter is the open cone spanned by $S_{K_J}^+$ (and furthermore equals to the almost Kähler cone when J is good generic). Hence $[C] = \sum a_i [C_i]$ where $a_i > 0$ and $[C_i] \in S_{K_J}^+$. Notice we can choose a_i to be rational numbers since all the classes here are rational (in fact, integral). Furthermore, as noted in [13], any class is in $S_{K_J}^+$ is Cremona equivalent to one of the following classes

- (1) $H, 2H,$
- (2) $(n+1)H - nE_1, n \geq 1,$
- (3) $(n+1)H - nE_1 - E_2, n \geq 1.$

Here Cremona equivalence refers to the equivalence under the group of diffeomorphisms preserving the canonical class K_J . It is easy to check that each of them could be written as sum of classes of rational curves with square 0 or 1. This implies extremal rays (with $K \cdot [C] < 0$) have to be spanned by rational curves with $-3 \leq K \cdot [C] < 0$. There are countably many such classes, since there are countable many -1 curve classes. This finishes the proof of the first statement of our cone theorem on extremal rays.

For the finiteness statement, it makes non-trivial sense only when $M = \mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ with $k \geq 9$. This is because there are finitely many irreducible curves with $K \cdot [C] < 0$. If M is not rational or ruled, they are -1 rational curves E_i where M . If M is ruled they are $E_i, T - E_i$ or T if it is minimal. If $M = \mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ with $k < 9$, they are -1 rational curves (there are finitely many when $k < 9$), $H - E$ (when $k = 1$) and H (when $k = 0$).

Now we are amount to show that when $k \geq 9$ there are only finitely many -1 rational curve classes with bounded symplectic energy $[\omega] \cdot E < \frac{1}{\epsilon}$. Since being symplectic is an open condition, $[\omega] - \delta H$ is still a class of symplectic form when $\delta > 0$ is small. Moreover E is always represented by a (possibly reducible) symplectic surface since $SW(E) \neq 0$. Hence $[\omega] \cdot E < \frac{1}{\epsilon}$ would

imply

$$H \cdot E \leq \frac{1}{\delta}([\omega] - \delta H) \cdot E + H \cdot E = \frac{1}{\delta}[\omega] \cdot E < \frac{1}{\epsilon\delta}.$$

On the other hand there are only finitely many classes $E = aH - \sum c_i E_i$ with $E^2 = -1$ and $a > 0$ is bounded from above. Especially, the finiteness statement implies the extremal rays $\mathbb{R}^+[L_i]$ are discrete.

To conclude our proof, we first see that $\overline{A}_{K_J}(M) + \sum \mathbb{R}^+[L_i] \subset \overline{A}_J(M)$ is a closed convex cone. This is because the above discussion implies $\mathbb{R}^+[L_i]$ can only have accumulate points in $\overline{A}_{K_J}(M)$. On the other hand, to show $\overline{A}_J(M) \subset \overline{A}_{K_J}(M) + \sum \mathbb{R}^+[L_i]$, we only need to show the inclusion for all classes $e \in \overline{A}_J(M)$ with $K \cdot e < 0$. This is exactly what we have proved. \square

Remark 2.5. *What we have proved is actually a slightly stronger version of cone theorem:*

$$\overline{A}_J^{\mathbb{Q}}(M) = \overline{A}_{K_J}^{\mathbb{Q}}(M) + \sum \mathbb{Q}^+[L_i],$$

where $\overline{A}_J^{\mathbb{Q}}(M) = \overline{A}_J(M) \cap H^2(M, \mathbb{Q})$ and $\overline{A}_{K_J}^{\mathbb{Q}}(M) = \overline{A}_{K_J}(M) \cap H^2(M, \mathbb{Q})$.

The next example shows that it is not true that any Seiberg-Witten non-trivial class of nonzero J -genus is an integral combination of curve classes.

Example 2.6. *Let $M = \mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}$. Then $SW(-K) = 1$ because of $SW(2K) = 0$ and the wall crossing formula. However, $-K$ cannot be written as $m_1[C_1] + m_2[C_2]$ with $m_1 m_2 \neq 0$ and $m_i \in \mathbb{Z}$ such that $SW([C_i]) \neq 0$ for $i = 1, 2$. This is because the Seiberg-Witten is a deformation invariant. Especially, $[C_1]$ and $[C_2]$ are classes of (possibly reducible) symplectic surfaces in a Del Pezzo surface, i.e. a 4-manifold where $-K$ is the class of the symplectic form. Hence $(-K) \cdot [C_i] \geq 1$ since all the classes are integral. It contradicts to $(-K) \cdot (m_1[C_1] + m_2[C_2]) = (-K)^2 = 1$.*

On the other hand, it has the following decomposition with rational coefficients:

$$-K = \frac{1}{2}(6H - 3E_1 - 2E_2 - \cdots - 2E_8) + \frac{1}{2}E_1.$$

We have the following more specific description of the extremal rays.

Proposition 2.7. *Let (M, J) be a tamed almost complex 4-manifold. An irreducible curve C is an extremal rational curve as in Theorem 2.4 if and only if*

- (1) C is a -1 rational curve;
- (2) M is a minimal ruled surface or $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, and C is a fiber;
- (3) $M = \mathbb{C}P^2$ and C is a projective line.

Proof. In the above proof of Proposition 2.4, we see that when M is not rational or ruled, all the extremal rays for non-minimal manifolds are -1 classes. For irrational ruled surfaces, our proof shows that any extremal curve class C has $-2 \leq K \cdot C < 0$. The $K \cdot C = -2$ case is when C is the

fiber class T . However when M is non-minimal, $T = E + (T - E)$ is not extremal.

Similarly for rational surfaces, we know that any square 1 (resp. 0) sphere class is Cremona equivalent to H (resp. $H - E$). When $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$, $k \geq 2$, we know both classes could be decomposed into two classes with non-trivial Seiberg-Witten invariant: $H = E + (H - E)$, $H - E = E' + (H - E - E')$. Hence they are not extremal.

When $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, $H = E + (H - E)$ is the sum of two classes with non-trivial Seiberg-Witten invariant. But it is possible that $H - E$ is the only extremal ray. In this case, the effective class E degenerates as $n(H - E) + ((n + 1)E - nH)$. Then $(n + 1)E - nH$ is the class of a $-(2n + 1)$ section for the ruled surface. It corresponds to Hirzebruch surfaces \mathbb{F}_{2n+1} when J is complex. \square

The following lemma gives our information on how a general extremal ray of $A_J(M)$ could be.

Lemma 2.8. *If C is an irreducible curve with $C^2 < 0$, then $[C]$ spanned an extremal ray of the curve cone $A_J(M)$.*

Proof. If $\mathbb{R}^+[C]$ is not extremal, then $[C] = \sum a_i [C_i]$ where $a_i > 0$ and C_i are irreducible curves whose homology classes are not on $\mathbb{R}^+[C]$. For those C_i , $[C] \cdot [C_i] \geq 0$. Hence we have the following contradiction

$$0 \leq \sum a_i [C] \cdot [C_i] = [C] \cdot [C] < 0.$$

\square

The following rationality theorem is originally used in algebraic geometry to prove the cone theorem. It is well known that the statement of cone theorem implies the statement of the rationality theorem.

Proposition 2.9. *Let (M, J) be a tamed almost complex 4-manifold for which K_J is not J -nef (i.e. it pairs non-negatively with the curve cone $A_J(M)$). Let ω be an almost Kähler form on (M, J) with $[\omega] \in H^2(M, \mathbb{Q})$. Define the nef threshold of $[\omega]$ by*

$$t_0 = t([\omega]) = \sup\{t \in \mathbb{R} : tK_J + [\omega] \text{ is } J\text{-nef}\}.$$

Then the nef threshold is a rational number.

Proof. It is easy to see that

$$t_0 = \sup \frac{L_i \cdot [\omega]}{-K_J \cdot L_i}.$$

If there are finitely many extremal curves L_i , then t_0 is a rational number by this formula. In general, since K_J is not J -nef, there exists a small number ϵ such that $\frac{1}{\epsilon}K + [\omega]$ is not J -nef which pairs negatively with only finitely many extremal curves. Hence t_0 is the supreme of $\frac{L_i \cdot [\omega]}{-K_J \cdot L_i}$ for these finitely many L_i , which has to be a rational number. In fact, in this case, t_0 is an

integer since the situation of the second statement of Theorem 2.4 happens only when all L_i are -1 rational curves.

Moreover, when $[\omega] \in H^2(M, \mathbb{Z})$, then the denominator of t_0 is no greater than 3. It is not an integer only when the last two cases of Proposition 2.7 happen. \square

Corollary 2.10. *Let $M = N \# \overline{\mathbb{C}P^2}$ be a non-minimal symplectic 4-manifold which is not diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Then for any tamed J on M , there exists at least one smooth J -holomorphic -1 rational curve.*

More precisely, for any exceptional class E , we have a decomposition $E = \sum a_i [C_i]$ with $0 < a_i \in \mathbb{Q}$ and C_i irreducible curves, such that there is at least one -1 rational curves C_i .

Proof. By the assumption, there is at least one -1 rational curve class $E \in H_2(M, \mathbb{Z})$. If there is an irreducible J -holomorphic subvariety in class E , then we are done since it will be smooth by adjunction inequality. If not, since the curve cone is a convex cone, the class is written as $E = \sum a_i [C_i]$ where all C_i are irreducible J -holomorphic subvarieties and all $[C_i]$ are extremal curve classes. Furthermore, $a_i \in \mathbb{Q}$ since all the classes E and $[C_i]$ are in $H_2(M, \mathbb{Z})$.

Since $K \cdot E = -1 < 0$, we know there is at least one C_i (say C_1) such that $K \cdot [C_1] < 0$. By the cone theorem, this irreducible C_1 has to be a rational curve with $-3 \leq K \cdot [C_1] < 0$. Moreover, by comparing to the list in Proposition 2.7, C_1 has to be the class of a -1 rational curve. \square

Notice that the statement that $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ has at least one smooth -1 rational curve is first proved in [26]. Actually, it shows that a class with minimal symplectic energy is such a class. However, our proof gives more precise result. The second statement of the above corollary is crucial for our later applications.

It is interesting to compare our picture here for a general tamed almost complex structure to the bend-and-break in algebraic geometry. The bend-and-break technique in algebraic geometry starts with an irreducible curve C' with $K \cdot [C'] < 0$. One chooses a normalization $f : C \rightarrow M$ of C' . Then it contains two parts. The first, the “bend” part, is to compose the normalization with automorphisms of C , possibly in characteristic p when $g(C) > 1$, such that $-K \cdot f'(C) - g(C) \dim_{\mathbb{C}} M > 0$. This would guarantee one could deform curves in a class which is a multiple of $[C']$. The second, the “break” part, shows that this family must degenerate to $f''(C) + (\text{sum of rational curves})$.

Our argument is sort of a reverse process. We show that all the extremal rays with negative K pairing are spanned by rational curves. And thus a higher multiple of the curve class C with $K \cdot C < 0$ will degenerate to a reducible curve with at least one extremal ray as one of its irreducible components.

In general, it is not true that we always have a reducible curve in class $[C]$ if C is an irreducible J -holomorphic curve of positive genus such that $K \cdot [C] < 0$ as we have seen in Example 2.6. Here is an example for ruled surfaces.

Example 2.11. *Let M be the non-trivial S^2 bundle over Σ_h . Let U be the class of a section with $U^2 = 1$. For $a = \lceil \frac{h-2}{2} \rceil$, we have $SW(U + aT) \neq 0$ and $SW(U + (a-1)T) = 0$. Thus for a generic tamed almost complex structure, we do not have reducible curves in class $U + aT$. However, as shown in the proof of Theorem 2.4, we do have curves in class $lU - T$ when $l > 2$. Hence there is always a reducible subvariety in class $l(U + aT)$.*

On the other hand, as shown in our proof, the class $[C]$ itself contains reducible curves in many circumstances. We also have the following slight variant of the well known fact that rational or ruled 4-manifolds are symplectically uniruled.

Proposition 2.12. *Let (M, J) be a tamed almost complex 4-manifold with a J -ample anti-canonical class $-K$ (i.e. $-K$ pairs positively with all curves). Then M contains a rational curve. In fact, through every point of M there is a rational curve C such that*

$$0 < -K \cdot [C] \leq 3.$$

Proof. By Taubes' theorem [28], when $-K$ is ample, M has to be rational or ruled. Furthermore, it cannot be irrational ruled, otherwise $(-K) \cdot (U + aT) \leq 0$ and $SW(U + aT) \neq 0$ where $a = \lceil \frac{h-2}{2} \rceil$ when M is the non-trivial S^2 bundle over Σ_h and $a = \lceil \frac{h-1}{2} \rceil$ when $M = S^2 \times \Sigma_h \#_k \overline{\mathbb{C}P^2}$. Hence, M is rational and we choose homology basis such that $K = -3H + E_1 + \dots + E_k$.

Since $K \cdot [C] < 0$ for all irreducible curves, then either the curve is a -1 rational curve or it has $(K - [C]) \cdot [D] < 0$ for any irreducible curve (and then any subvariety) D . In both cases, $\dim_{SW}([C]) = C^2 - K \cdot [C] = 2g - 2 - 2K \cdot [C] \geq 0$. Hence $SW([C]) \neq 0$ by Lemma 2.2 and the wall-crossing formula.

Since $SW([C]) \neq 0$ and H is represented by an irreducible J' -holomorphic curve with positive self-intersection for a generic J' , by Lemma 2.2, $H \cdot [C] \geq 0$ for all curves C . In other words, H is J -nef when $-K$ is J -ample. Then by Theorem 1.5 of [20], we know any irreducible component C_i of a subvariety in class H is a rational curve with $0 < -K \cdot [C_i] < -K \cdot H = 3$. Then the conclusion follows since there is a subvariety in class H passing through any given point. \square

3. $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$

This section serves as a link between Sections 2 and 4. We first recall some general results for the homology classes of irreducible subvarieties on $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$. Then we apply them to give an explicit description of the negative curves on $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$. In particular, we show there are at least

two smooth -1 rational curves for any tamed almost complex structure on it in Theorem 3.6. A full description of the curve cone is given in Theorem 3.10. The information on the curve cone helps us to obtain a Nakai-Moishezon type duality, Theorem 3.12.

3.1. The curve cone and the K -symplectic cone. The K -symplectic cone for a class $K \in H^2(M; \mathbb{Z})$ introduced in [16]:

$$(2) \quad \mathcal{C}_{M,K} = \{e \in H^2(M; \mathbb{R}) | e = [\omega] \text{ for some } \omega \text{ with } K_\omega = K\}.$$

Here K_ω is the symplectic canonical class of ω . Suppose $\mathcal{C}_{M,K}$ is non-trivial, $[\omega] \in \mathcal{C}_{M,K}$ and $b^+(M) = 1$, by Theorem 3 in [16],

$$(3) \quad \mathcal{C}_{M,K} = \{e \in \mathcal{FP}(K) | e \cdot E > 0, E \in \mathcal{E}_K\}.$$

Here $\mathcal{FP}(K)$ is the connected component of $\mathcal{P} = \{e \in H^2(M, \mathbb{Z}) | e^2 > 0\}$ containing $[\omega]$. Notice by the light cone lemma, both $\mathcal{FP}(K)$ and $\mathcal{C}_{M,K}$ are convex cones. Recall the following statement which is called the light cone lemma in the literature, which is in the guise of Cauchy-Schwartz inequality. The cone of elements with positive squares have two components. The forward cone means one of the two connected components containing a given element with positive square. In our applications, it is usually a class of symplectic form.

Lemma 3.1 (light cone lemma). *For the light cone of signature $(1, n)$ ($n \neq 0$), any two elements in the forward cone have non-negative dot product. Especially, if the dot product is zero then the two elements are proportional to each other.*

We will state a structural description of the K -symplectic of rational surfaces, which might be known for experts. Without loss, we suppose $K = -3H + \sum_i E_i$.

Proposition 3.2. *Let $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$.*

- (1) *When $k < 9$, the K -symplectic cone is a cone over a polytope whose corners are the classes of the symplectic spheres with canonical class K and self-intersection 0 or 1.*
- (2) *When $k \geq 9$, all the extremal rays \mathbb{R}^+e of the K -symplectic cone having $K \cdot e < 0$ are generated by the classes of the symplectic spheres with canonical class K and self-intersection 0 or 1.*

Proof. We first assume $k < 9$. The K -symplectic cone is a polytope follows from Proposition 2.7(1) in [9]. In fact, the K -symplectic cone is a P -cell when $k < 9$, and a P -cell has no round boundary. One could also see Section 5.4 of [19] for an overview of the results in [9] using notations similar to this paper's. For a more direct argument, see [24].

We now show that the corners of the polytope, which correspond to the extremal rays of the K -symplectic cone, are those classes which can be represented by the symplectic spheres with canonical class K and self-intersection 0 or 1. This statement follows from the light cone lemma. The corners of

the polytope are the intersection of several hyperplanes determined in our situation by classes in \mathcal{E}_K . Recall the K -symplectic cone is a cone over a polytope implies that the corners of the polytope are in $\overline{\mathcal{P}}$.

There are two types of corners. If the classes determine the hyperplanes around the corner ray \mathbb{R}^+e are orthogonal to each others, without loss we can assume them to be E_1, \dots, E_l . We know $l = k$ otherwise the intersection would include any classes $aH - bE_k$ with $a \geq b > 0$, hence is not a ray. But when $l = k$, the primitive class corresponding to the corner is a class Cremona equivalent to H , *i.e.* the class of a symplectic sphere with self-intersection 1.

If two of the classes determine the hyperplanes around a corner are not orthogonal, say E and E' (both in \mathcal{E}_k), then the corner class e is also orthogonal to $E + E'$ which is of non-negative self-intersection. By the light cone lemma, e is proportional to $E + E'$ and $(E + E')^2 = 0$. The equality happens only if $E \cdot E' = 1$ and $(E + E')^2 = 0$. The equality happens only if $E \cdot E' = 1$ and $E + E'$ is the class of a symplectic sphere of self-intersection 0 (since $K \cdot (E + E') = -2$). If we choose e to be primitive, then $e = E + E'$.

When $k \geq 9$, by Proposition 2.7 of [9], the intersection $\mathcal{C}_{M,K} \cap \{e \in H^2(M; \mathbb{R}) \mid K \cdot e \leq 0\}$ is a P -cell which has no round boundary. Hence, the extremal rays with $K \cdot e < 0$ corresponding to intersections of the ordinary walls (*i.e.* the hyperplanes determined by classes in \mathcal{E}_K rather than the canonical class K). Then we have exactly the same two types of corners, and the same argument applies. \square

Remark 3.3. *The K -symplectic cone has round boundary when $k > 9$. For examples, one can take the anti-canonical class $e = -K$. The class $e \notin \mathcal{C}_{M,K}$ when $k > 9$ since $e^2 < 0$ but $e \cdot E = 1 > 0$ for all $E \in \mathcal{E}_K$.*

Moreover, one can indeed get an open set of round boundary. Since $\mathcal{C}_{M,K} \neq \emptyset$, we choose an open set B of it. A class $-K + a_\omega[\omega]$, where $a_\omega > 0$ and $[\omega] \in B$, is on the round boundary of $\mathcal{C}_{M,K}$ if a_ω solves $(-K + a_\omega[\omega])^2 = 0$ (since $(-K + a_\omega[\omega]) \cdot E > 0$ for all $E \in \mathcal{E}_K$). This is a quadratic equation and such a_ω exists since $K^2 \cdot [\omega]^2 < 0$.

For a detailed discussion on the round boundary, see the new edition of [23].

The K -symplectic cone can be used to restrict the classes of negative square in the curve cone. The following lemma is simple but also very useful.

Lemma 3.4. *Suppose $b^+(M) = 1$. A cohomology class is in the curve cone only if it is positive on some extremal ray of the K -symplectic cone.*

Proof. Assume the class e is in the curve cone $A_J(M)$ for some tamed almost complex structure J with $K_J = K$. Let ω be a symplectic form taming J . Then $e \cdot [\omega] > 0$. Since $\mathcal{C}_{M,K}$ is convex, there is an extremal ray pairing positively with e . \square

The next lemma is on the constraints of the curve classes provided by the adjunction inequality.

Lemma 3.5. *Suppose a class $B = \alpha H + \sum \beta_i E_i$ has an irreducible curve representative.*

If $\alpha > 0$ then $|\beta_i| \leq |\alpha|$ for each i , and $|\alpha| = |\beta_i|$ only when $\alpha = -\beta_i = 1$.

If $\alpha = 0$, then $B = E_i - \sum_j E_{k_j}$.

If $\alpha < 0$, then $|\beta_i| \leq |\alpha| + 1$, and $|\beta_i| = |\alpha| + 1$ only if $\beta_i = -\alpha + 1$.

Proof. By adjunction formula, and the fact that $\gamma^2 + \gamma \geq 0$ for any integer γ , we have

$$(\alpha - 1)(\alpha - 2) \geq \beta_i(\beta_i + 1).$$

Then all the conclusions are clear when $\alpha \neq 0$.

For $\alpha = 0$, we first get that $\beta_i = 0, -2$ or ± 1 . However when $\alpha = 0$, some β_i is non-negative. Otherwise, J will not be tamed by lemma 3.4. Then $B = E_i - \sum_j E_{k_j}$ holds since $\sum \beta_i(\beta_i + 1) \leq 2$ by adjunction formula. \square

3.2. Curves on $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$. We have shown that there is at least one smooth J -holomorphic -1 rational curve for any tamed J on non-minimal symplectic manifold except for $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in Corollary 2.10. Now, we will show that when $M = \mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$, we actually have at least two -1 sphere classes. Notice it is not true when $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ for $k > 2$.

Theorem 3.6. *There are at least two smooth -1 J -holomorphic rational curves for any tamed J on $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$.*

Proof. First, there is at least one -1 smooth rational curve by Corollary 2.10. We first assume the class E_2 has such a smooth representative.

Lemma 3.7. *Suppose $M = \mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ and the class E_2 is the class of a smooth -1 rational curve. If a class $A = aH + bE_1 + cE_2 \neq E_2$ with $a \leq 0$ has an irreducible curve representative, then*

(i) $b > 0, a = 1 - b$ and $c = 0$ or -1 .

(ii) A is a sphere class.

(iii) $A \cdot A < 0$.

(iv) A is the only such class.

Proof. First $c \leq 0$ since $A \cdot E_2 = -c \geq 0$.

First, the set $\mathcal{E}_K = \{E_1, E_2, H - E_1 - E_2\}$. It could be checked directly by adjunction formula, or from section 4 (in particular, Lemma 4.1 and Proposition 4.4). Then by (3), the extremal rays of the K -symplectic cone are spanned by

$$H, \quad H - E_1, \quad H - E_2.$$

As $a \leq 0$ and $c \leq 0$, we have $A \cdot H \leq 0$ and $A \cdot (H - E_2) \leq 0$. Therefore by Lemma 3.4 $A \cdot (H - E_1)$ is strictly positive. This means that $a + b > 0$, i.e. $b > -a \geq 0$.

By the adjunction formula

$$(4) \quad (a - 1)(a - 2) \geq b(b + 1) + c(c + 1).$$

The only possibility is as claimed in (i), $A = (1 - b)H + bE_1$ or $A = (1 - b)H + bE_1 - E_2$ with $b \geq 1$. Items (ii) and (iii) are then direct to check.

For (iv), suppose $A' = (1 - b')H + b'E_1$ or $(1 - b')H + b'E_1 - E_2$ is another such class. Then $A \cdot A' = 1 - (b + b')$ or $-(b + b')$ is negative. \square

Lemma 3.8. *Suppose $M = \mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ and the class E_2 is the class of a smooth -1 rational curve. If $A = (1 - s)H + sE_1$ or $(1 - s)H + sE_1 - E_2$ with $s \geq 1$ is in the curve cone, then a class $D = H + vE_1 + wE_2$ is in the curve cone only if $v > -1$ or $v = -1, w \geq -1$.*

Proof. Since $s \geq 1$, D must be of the form $pA + \sum B_i$, where $B_i = \alpha_i H + \beta_i E_1 + \gamma_i E_2$ and $p \geq 0$. By Lemma 3.7 (iv), we have $\alpha_i > 0$ or $B_i = E_2$. Then pairing with H , we have

$$1 = p(1 - s) + \sum \alpha_i.$$

Now pairing with E_1 , we have

$$v = ps + \sum \beta_i = p - 1 + \sum (\alpha_i + \beta_i) \geq -1$$

by Lemmas 3.5 and 3.7. Moreover, $v = -1$ only if $p = 0$, and by Lemma 3.5, we have

$$w = \sum \gamma_i \geq -\sum \alpha_i = -1.$$

\square

Corollary 3.9. *Suppose E_2 is the class of a smooth -1 rational curve. Then so does either the class $H - E_1 - E_2$ or the class E_1 . And in the latter case, both E_1 and $H - E_1 - E_2$ are the classes of smooth -1 rational curves.*

Proof. Suppose E_2 has an embedded representative and E_1 does not. Note the class $H - E_1 - E_2$ is in the curve cone since $SW(H - E_1 - E_2) \neq 0$. By Corollary 2.10, there is a -1 rational curve C_i in a decomposition $H - E_1 - E_2 = \sum_i a_i [C_i]$ (in other words, an irreducible component of a \mathbb{R} -variety in class $H - E_1 - E_2$). By our assumption, this class cannot be E_1 . If this class is $H - E_1 - E_2$, we are done.

If this class is E_2 , then $H - E_1 - lE_2$, with $l > 1$, is in the curve cone. If we have an irreducible curve in class $(1 - s)H + sE_1$ or $(1 - s)H + sE_1 - E_2$ with $s \geq 1$, it will contradict to the Lemma 3.8. So all irreducible curves C have $a = C \cdot H > 0$. By Lemma 3.5, all such irreducible curves $aH - b_1 E_1 - b_2 E_2$ have $b_2 \leq a$. Hence $l \leq 1$.

If both E_1 and E_2 have embedded representatives, same argument shows that neither can appear in the decomposition of $H - E_1 - E_2$. \square

Now, to finish the proof of Theorem 3.6, we are left with case that the class $H - E_1 - E_2$ has an embedded representative.

By Corollary 2.10, there will be a -1 rational curve as an irreducible component of the \mathbb{R} -subvariety representing class E_1 or E_2 . Suppose that there is no irreducible curve with non-positive H coefficient. Then $H - E_1 - E_2$ cannot appear as the class of such -1 rational curve. Thus the -1 class

component in E_1 is either E_1 or E_2 . Hence in this situation, there are at least two -1 rational curves.

Thus we assume that there is an irreducible curve in class $A = aH + bE_1 + cE_2$ with $a \leq 0$.

Since $a \leq 0$, either $a + b > 0$ or $a + c > 0$ by Lemma 3.4.

Without loss, we assume that $a + b > 0$. We will show that in this case E_1 or E_2 must have an embedded representative.

First by adjunction formula

$$(a - 1)(a - 2) \geq b(b + 1) + (c^2 + c).$$

On the other hand $b \geq -a + 1 > 0$. Hence the only possibility for the adjunction holds would be $a = -b + 1$, $c = 0$ or -1 and $g = 0$. Then $A = (1 - b)H + bE_1$ or $A = (1 - b)H + bE_1 - E_2$ with $b \geq 1$.

If E_1 and E_2 do not have irreducible representative, then $H - E_1 - E_2$ is the only class of extremal irreducible curve with $K \cdot C < 0$ as shown in Corollary 2.10. We now look at irreducible curves with $K \cdot C \geq 0$. First let $[C] = aH - b_1E_1 - b_2E_2$, $a > 0$. Then $K \cdot C \geq 0$ implies

$$0 < 3a \leq b_1 + b_2.$$

But by local positivity of intersections, $C \cdot (H - E_1 - E_2) \geq 0$, which is $a \geq b_1 + b_2$. It is a contradiction. Hence $a \leq 0$ and the curves classes are calculated as above. Hence E_i will be a linear combination of these classes and $H - E_1 - E_2$. If we write a class as $aH + b_1E_1 + b_2E_2$, then all the above classes will contribute non-positively to $2a + b_1 + b_2$. However E_i has positive $2a + b_1 + b_2$. This is a contradiction. Hence there is an irreducible curve in class E_1 or E_2 . \square

The above discussion actually gives the following description of the curve cone. By Theorem 3.6, there is always an irreducible J -holomorphic curve in class E_1 or E_2 . Hence, without loss, we could assume E_2 has an irreducible representative.

Theorem 3.10. *Let J be a tamed almost complex structure on $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$ such that there is a smooth J -holomorphic curve in the class E_2 . Then the curve cone $A_J(M)$ is generated by 3 classes. They are either*

$$\alpha = (1 - s)H + sE_1, \quad \beta = E_2, \quad \gamma = H - E_1 - E_2,$$

or

$$\alpha = (1 - s)H + sE_1 - E_2, \quad \beta = E_2, \quad \gamma = H - E_1 - E_2,$$

where $s \geq 1$.

Proof. First by Lemma 3.9, there is always an irreducible curve in class $H - E_1 - E_2$. By the argument in the last paragraph of the proof of Theorem 3.6, curve classes C with $C \cdot H > 0$ is always spanned by $H - E_1 - E_2$, one of E_i say E_2 and another irreducible curve with non-positive pairing with H . When E_2 has irreducible representative, the last curve class is $(1 - s)H + sE_1$

or $(1-s)H + sE_1 - E_2$ with $s \geq 1$. By Lemma 3.7 (iv), such a curve is unique. This completes our proof. \square

Now we can study the ≥ 0 -dual of the curve cone $A_J(M)$.

If A_J is generated by $E_1, E_2, H - E_1 - E_2$, then its ≥ 0 -dual is generated by $H, H - E_1, H - E_2$.

Let us assume that E_2 is irreducible. We discuss the two cases in Theorem 3.10. In the first case, the ≥ 0 -dual of A_J is generated by

$$A = sH - (s-1)E_1, \quad B = H - E_1, \quad C = sH - (s-1)E_1 - E_2.$$

In the second case, the ≥ 0 -dual of A_J is generated by

$$A = sH - (s-1)E_1, \quad B = H - E_1, \quad C = (s+1)H - sE_1 - E_2.$$

In both cases, A and C are in $S_{K_J}^+$ and B is approximated by the sequence $pH - (p-1)E_1 - E_2, p \rightarrow \infty$ in $S_{K_J}^+$.

All the above actually shows the following

Proposition 3.11. *For any tamed J on $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$, we have*

$$\mathcal{S}_J = \mathcal{P}_J.$$

Recall that $\mathcal{P}_J = A_J^{\vee, >0}(M) \cap \mathcal{P}$. The spherical cone \mathcal{S}_J here is defined to be the interior of the convex cone generated by big J -nef classes (*i.e.* J -nef classes with positive square) in S_{K_J} if it is of dimension 3.

3.3. Nakai-Moishezon type theorem for almost Kähler structure on $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$. With Proposition 3.11 in hand, we can establish the Nakai-Moishezon and Kleiman type theorems for almost Kähler J on $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$.

Theorem 3.12. *Let $M = \mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$. For any almost Kähler J , the J -compatible cone $\mathcal{K}_J^c(M)$ is dual to the J -curve cone $A_J(M)$, *i.e.**

$$\mathcal{K}_J^c = \mathcal{P}_J = A_J^{\vee, >0}(M).$$

Proof. It is clear that $\mathcal{K}_J^c \subset \mathcal{P}_J$. If J is almost Kähler, we have $\mathcal{S}_J \subset \mathcal{K}_J^c$ (Lemma 5.18 in [19]). By Proposition 3.11, we have

$$\mathcal{K}_J^c = \mathcal{P}_J = \mathcal{S}_J.$$

The second equality $\mathcal{P}_J = A_J^{\vee, >0}(M)$ holds because the classes A, B, C and thus their positive combinations all have non-negative squares. \square

We remark that the techniques in [19] to construct almost Kähler form for a tamed J fail in this situation.

First we need to construct a Taubes current. Here a current is a differential form with distribution coefficients. Hence it represents a real cohomology class when pairing with smooth closed forms in the weak sense. A Taubes current is a closed, positive J -invariant current Φ , which satisfies

$$k^{-1}t^4 \leq \Phi(if_{B_t(x)}\sigma \wedge \bar{\sigma}) < kt^4.$$

Here σ denotes a point-wise unit length section of $T^{1,0}M|_{B_t(x)}$. The usual technique for the construction is to integrate certain part of the moduli space of the subvarieties in a J -ample class e , *i.e.* a cohomology class pairing positively with any curve classes, with $g_J(e) = 0$. However in general, as in current situation, we do not have any such classes. We may only have big J -nef spherical classes. In this situation, we are still able to produce a weak version of Taubes current in class e : a closed, non-negative J -invariant current Φ_e , satisfying

$$0 \leq \Phi_e(ief_B\sigma \wedge \bar{\sigma}) < kt^4.$$

It will vanish along the vanishing locus $Z(e)$, *i.e.* the union of irreducible subvarieties D_i such that $e \cdot D_i = 0$. But over any 4-dimensional compact submanifold with boundary K of the complement $M(e) = M \setminus Z(e)$, it is a Taubes current with the constant $k > 1$ depending only on K .

If we have sufficiently many big J -nef classes, we could produce genuine Taubes currents by the following Proposition 5.7 in [19].

Proposition 3.13. *Let e_i be big J -nef classes in S_{K_J} and the zero locus of e_i is denoted by Z_i . If $\cap Z_i = \emptyset$, then there is a Taubes current in the class $e = \sum_i a_i e_i$, with $a_i > 0$.*

Finally, we apply the following regularization result of [30] (see also [31]) to obtain an almost Kähler form in the class e .

Theorem 3.14. *In a 4-manifold M with $b^+(M) = 1$, if we have a Taubes current T , then there is an almost Kähler form ω , s.t. $[\omega] = [T]$.*

Hence to construct an almost Kähler form by the subvariety-current-form method, we are reduced to prove that there exist big J -nef classes e_i in S_{K_J} , such that the intersection of the zero locus $\cap Z(e_i) = \emptyset$. We claim it is impossible if our J is assumed only to be tamed. In the below, α, β, γ are those classes in Theorem 3.10. The main point is there is no class $e \in S_J^+$ such that $Z(e) = \alpha$ by simple homological calculation. Since $\beta \cdot \gamma = 1$, we have $\beta \cap \gamma \neq \emptyset$. Because of the above observation, any class $e \in S_J^+$ will have $\beta \cap \gamma \subset Z(e)$. Hence $\beta \cap \gamma \subset \cap Z(e_i)$, which is then not empty.

4. CONFIGURATIONS OF NEGATIVE CURVES ON RATIONAL AND RULED SURFACES

Almost complex structures are different from complex structures at blowing up and down. More precisely, when we have an irreducible holomorphic -1 sphere, we can always blow it down by Castelnuovo's criterion. However, generally we cannot blow down a smooth J -holomorphic -1 rational curve for a tamed almost complex structure J .

In this section, we study the negative curves in a tamed almost complex rational or ruled 4-manifolds, which might not be mentioned explicitly in each statement.

4.1. Negative curves on $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ with $k \leq 9$. In this subsection, we will study the negative curves on rational surfaces. We will show that they have to be spheres for $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ with $k \leq 9$. This will enable us to determine the curve cone and show that the configurations of negative curves for tamed almost complex structures are all realized by complex structures (Theorem 1.2). Note that the case of $S^2 \times S^2$ and $\mathbb{C}P^2 \# 2 \overline{\mathbb{C}P^2}$ have been done in [19], $\mathbb{C}P^2 \# 2 \overline{\mathbb{C}P^2}$ is done in section 3.

We first take a look at the irreducible curve classes $C = aH + \sum b_i E_i$ with $C^2 < 0$ and $a < 0$.

Lemma 4.1. *Let $M = \mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$. If $C = aH + \sum b_i E_i$ with $a < 0$ is represented by an irreducible curve, then*

- $C = -nH + (n+1)E_1 - \sum_{k_j \neq 1} E_{k_j}$ up to diffeomorphism.
- Or $C = f^*C'$, where f is a diffeomorphism and C' is a class with $a' > 0$.

Proof. The proof is eventually similar to that of Lemma 3.7. As we suppose our canonical class $K = -3H + \sum E_i$, there are two types of the generators of extremal rays of the K -symplectic cone. The first type is the classes F with $F^2 = 1$ which can be represented as a sphere. By [13], those are Cremona equivalent to H . The second type of classes are those Cremona equivalent to $H - E_1$.

Let us first suppose that C pairs non-positively with all the classes of the first type. By Lemma 3.4, at least one of these classes equivalent to $H - E_1$ pairs positively with C . If we suppose it is $H - E_1$, then we know that $b_1 > -a > 0$. Then by the adjunction formula $(a-1)(a-2) \geq \sum b_i(b_i+1)$. Thus, the only possibility is as claimed, $C = -nH + (n+1)E_1 - \sum_j E_{k_j}$ which has $g_J(C) = 0$.

If $C \cdot F > 0$ for some F of the first type, then we can first change the class F to H by a diffeomorphism f preserving the canonical class. The class C changes to C' at the same time and thus $C' \cdot H > 0$. Thus C' is a class with $a' > 0$ and C is pull-back of it by a diffeomorphism. \square

The latter case could happen. For example when $C = -H + E_1 + E_2 + E_3$, $F = 2H - E_1 - E_2 - E_3$. Then C is equivalent to $H - E_1 - E_2 - E_3$ after a Dehn twist along $H - E_1 - E_2 - E_3$. However, all -1 rational curve classes $C = aH + \sum b_i E_i$ have $a \geq 0$. This can be seen by applying Lemma 2.2 to the Seiberg-Witten nontrivial classes H and C .

The case when $a = 0$ is investigated in Lemma 3.5. The only possible curves are $E_i - \sum_{k_j \neq i} E_{k_j}$.

Now, let us take a look at the case of $a > 0$.

Proposition 4.2. *Suppose $M = \mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$, $k \leq 9$.*

- (1) *Then any irreducible curves C with $C^2 < 0$ are smooth spheres.*
- (2) *If $k \leq 8$, any irreducible curves with $C^2 \leq 0$ are smooth spheres.*

Proof. Let $C = aH - \sum b_i E_i$. The case when $a \leq 0$ is discussed above. The only undetermined case, the second bullet of Lemma 4.1 is reduced to the case of $a > 0$. Hence, we suppose $a \geq 1$ below.

Because $C^2 < 0$, we can suppose

$$(5) \quad c^2 + a^2 = b_1^2 + \cdots + b_k^2, \quad c \in \mathbb{R} \setminus \{0\}.$$

In other words, $c^2 = -C^2$.

If C is an irreducible curve and is not a sphere, then by adjunction formula,

$$(6) \quad c^2 + 3a \leq b_1 + \cdots + b_k.$$

Hence, by Cauchy-Schwartz inequality, we have $(c^2 + 3a)^2 \leq k(c^2 + a^2) \leq 9(c^2 + a^2)$, *i.e.*

$$6ac^2 + c^4 - 9c^2 \leq 0.$$

This inequality is possible only when $a = 1$. Then (6) becomes

$$b_1 + \cdots + b_k \geq 3 + c^2.$$

It contradicts to (5), which reads as

$$3 + c^2 \leq b_1 + \cdots + b_k \leq b_1^2 + \cdots + b_k^2 = 1 + c^2.$$

This contradiction shows that C should be a sphere.

We also notice that when $k \leq 8$, any irreducible curves with $C^2 \leq 0$ are spheres. This is because if $C^2 = 0$, formulae (5), (6) lead to the contradiction

$$ka^2 \geq (3a)^2.$$

□

Notice that the statement is sharp, in the sense that it is no longer true for $k \geq 10$ (resp. $k \geq 9$), since the anti-canonical class K could be the class of an elliptic curve with $K^2 < 0$ (resp. $K^2 \leq 0$).

It is true that these negative curves form the extremal rays of the curve cone.

Proposition 4.3. *The curve cone $A_J(M)$ of $(M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, J)$, $k < 9$, is a polytopic cone generated by the classes of spheres with non-positive self-intersections.*

Proof. By Proposition 4.2, we just need to prove that any irreducible curve C with $C^2 > 0$ cannot span an extremal ray of the curve cone $A_J(M)$.

We look at the class $n[C]$. We know that when $k \leq 8$, there are only finitely many elements in \mathcal{E}_K . This is because first there are finitely many classes in \mathcal{E}_K such that its pairing with H is non-negative (*e.g.* Proposition 4.4). Second, since there are finitely many classes F such that $F^2 = 1$ and $K \cdot F = -3$ when $k \geq 8$, the action of the Cremona group which gives the classes in the second bullet of Lemma 4.1 have all orbits finite (see [13]).

Now we assume the maximal possible pairing of C with elements in \mathcal{E}_K is l . We choose n large enough such that

$$(7) \quad n^2 C^2 - n(K \cdot [C] + 2l) - 2 > 0, \quad n^2 C^2 - 2n(K \cdot [C] + l) + 6 - k > 0.$$

This is possible, since the coefficient of the quadratic term is positive.

Now we claim $[C] = \frac{1}{n}((n[C] - E) + E)$, $E \in \mathcal{E}_K$, gives a decomposition with both classes having nontrivial Seiberg-Witten. Hence $[C]$ is not extremal.

First, we check the SW dimension

$$\begin{aligned} \dim_{SW}(n[C] - E) &= (n[C] - E)^2 - K \cdot (n[C] - E) \\ &= n^2 C^2 - nK \cdot [C] - 2n[C] \cdot E - 2 \\ &> 0 \end{aligned}$$

Take a symplectic form ω taming J . If $SW(K - (n[C] - E)) \neq 0$, then $[\omega] \cdot (K - (n[C] - E)) > 0$. We also have $(K - (n[C] - E))^2 > 0$ by the second inequality of (7). On the other hand, by our assumptions, $[\omega] \cdot [C] > 0$ and $C^2 > 0$. Hence by the light cone lemma, we have $(K - (n[C] - E)) \cdot [C] \geq 0$, which contradicts to the first inequality of (7). Hence we have $SW(K - (n[C] - E)) = 0$. By wall crossing,

$$|SW(n[C] - E)| = |SW(n[C] - E) - SW(K - (n[C] - E))| = 1.$$

Apparently $SW(E) = 1$. Hence $[C] = \frac{1}{n}((n[C] - E) + E)$ is not extremal.

Finally, since we have classified all the classes of irreducible curves with non-positive self-intersections when $k < 9$ in Proposition 4.1 and 4.4. Especially, there are finitely many such classes. Hence our conclusion follows. In particular, our curve cone has no round boundary. \square

This should be compared with Lemma 2.8.

Next, let us classify the negative irreducible curves with $a > 0$ on $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ with $k < 9$.

Proposition 4.4. *Let J be a tamed almost complex structure on $M = \mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$, $k < 9$, and $C = aH - \sum b_i E_i$ be an irreducible curve with $C^2 < 0$, $a > 0$. Then $[C]$ is one of the following:*

- (1) $H - \sum E_{k_j}$;
- (2) $2H - \sum E_{k_j}$;
- (3) $3H - 2E_m - \sum_{k_j \neq m} E_{k_j}$;
- (4) $4H - 2E_{m_1} - 2E_{m_2} - 2E_{m_3} - \sum_{k_j \neq m_i} E_{k_j}$;
- (5) $5H - E_{m_1} - E_{m_2} - \sum_{k_j \neq m_i} 2E_{k_j}$;
- (6) $6H - 3E_{m_1} - \sum_{k_j \neq m_1} 2E_{k_j}$.

Proof. Similar to Proposition 4.2, we have

$$\begin{aligned} c^2 + a^2 &= b_1^2 + \cdots + b_k^2, \\ -2 + c^2 + 3a &\leq b_1 + \cdots + b_k. \end{aligned}$$

Now, $(c^2 + 3a - 2)^2 \leq k(c^2 + a^2) \leq 8(c^2 + a^2)$ holds by Cauchy-Schwartz inequality. This can be written as

$$(8) \quad a^2 - 3a + (3c^2 - 9)a + (3a - 12)c^2 + c^4 + 4 \leq 0.$$

First let us assume $C^2 < 0$. The cases when $c^2 < 3$ (*i.e.* $-3 < C^2 < 0$) actually follow from the classification of possible -1 and -2 sphere classes, see for example [21]. More precisely, when $c^2 = 1$, the classification is obtained in Proposition 26.1, diagram (IV.8) of [21]. Especially, it contains our classes (3)-(6). When $c^2 = 2$, the classification is in diagram (IV.2) there. Let us reproduce the proof for readers' convenience. When $c^2 = 1$, we have

$$3a - \sum_{i=1}^r b_i = 1, \quad a^2 - \sum_{i=1}^r b_i^2 = -1, \quad r < 9.$$

By possibly adding some $b_i = 0$ and $b_9 = 1$, we are reduced to solve

$$3a - \sum_{i=1}^9 b_i = 0, \quad a^2 - \sum_{i=1}^9 b_i^2 = -2, \quad b_9 = 1.$$

Rewrite the second equation, we have

$$3a - \sum_{i=1}^9 b_i = 0, \quad \sum_{i=1}^9 (a - 3b_i)^2 = 18, \quad b_9 = 1.$$

In total there are three essentially different representations of 18 as a sum of 9 squares which are in the same residue class mod 3:

$$18 = 3^2 + 3^2 + 0^2 + \cdots + 0^2 = (\pm 2)^2 + (\pm 2)^2 + (\pm 2)^2 + (\mp 1)^2 + \cdots (\mp 1)^2.$$

Up to the order of b_i , the solutions $(a; b_1, \dots, b_9)$ are

$$(3b; b + 1, b - 1, b, \dots, b), \quad (3b \pm 2; b, b, b, b \pm 1, \dots, b \pm 1).$$

Notice one of b_i has to be 1, this gives the list in our statement.

Similarly, when $c^2 = 2$, our equations are reduced to

$$3a - \sum_{i=1}^9 b_i = 0, \quad \sum_{i=1}^9 (a - 3b_i)^2 = 18, \quad b_9 = 0.$$

Up to the order of b_i , we have the same general solutions, but now with one of b_i being 0. These will also lie in the list.

Now let us assume $C^2 \leq -3$, *i.e.* $c^2 \geq 3$. Then we have $a \leq 3$ by (8).

When $a = 1$ or 2 , then by adjunction, $b_i = 1$ or 0 . This corresponds to our classes (1) and (2).

When $a = 3$, we have

$$\sum (b_i - 1)b_i = 2.$$

Hence only one b_i could be 2 or -1 , others are 1 or 0. However, if $a = 3$ and b_i are ± 1 or 0, and $k < 9$, then $C^2 > 0$, a contradiction. Hence, it lies in class (3). \square

Notice all -1 curve classes could be realized by complex structures. Hence Proposition 4.4 says that all the negative curves with $a > 0$ are obtained from the -1 curves by further blow ups and one can obtain other negative curves by blowing up up to 6 more times. No other classes occur even for a tamed almost complex structure.

The method used in the above proof could be extended to non-negative curves as well. We will prove the case for $C^2 = 0$ in the following.

Proposition 4.5. *Let J be a tamed almost complex structure on $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$, $k < 9$, and $C = aH - \sum b_i E_i$ be an irreducible curve with $C^2 = 0$. Then $[C]$ is one of the following:*

- (1) $H - E_{m_1}$;
- (2) $2H - E_{m_1} - \cdots - E_{m_4}$;
- (3) $3H - 2E_{m_1} - E_{m_2} - \cdots - E_{m_6}$;
- (4) $4H - 2E_{m_1} - 2E_{m_2} - 2E_{m_3} - E_{m_4} - \cdots - E_{m_7}$;
- (5) $5H - E_{m_1} - 2E_{m_2} - \cdots - 2E_{m_7}$;
- (6) $6H - 3E_{m_1} - 3E_{m_2} - 2E_{m_3} - \cdots - 2E_{m_6} - E_{m_7} - E_{m_8}$;
- (7) $7H - 3E_{m_1} - \cdots - 3E_{m_4} - 2E_{m_5} - \cdots - 2E_{m_7} - E_{m_8}$;
- (8) $5H - 3E_{m_1} - 2E_{m_2} - \cdots - 2E_{m_4} - E_{m_5} - \cdots - E_{m_8}$;
- (9) $8H - 3E_{m_1} - \cdots - 3E_{m_7} - E_{m_8}$;
- (10) $4H - 3E_{m_1} - E_{m_2} - \cdots - E_{m_8}$;
- (11) $8H - 4E_{m_1} - 3E_{m_2} - \cdots - 3E_{m_5} - 2E_{m_6} - \cdots - 2E_{m_8}$;
- (12) $7H - 4E_{m_1} - 3E_{m_2} - 2E_{m_3} - \cdots - 2E_{m_8}$;
- (13) $9H - 4E_{m_1} - 4E_{m_2} - 3E_{m_3} - \cdots - 3E_{m_7} - 2E_{m_8}$;
- (14) $11H - 4E_{m_1} - \cdots - 4E_{m_7} - 3E_{m_8}$;
- (15) $10H - 4E_{m_1} - \cdots - 4E_{m_4} - 3E_{m_5} - \cdots - 3E_{m_8}$.

Proof. First, by Proposition 4.2, all these C are spheres.

We can still use (8). If $C^2 = 0$, then $c = 0$. Hence we have $0 < a \leq 11$. By Lemma 2.2, we have $b_i \geq 0$.

Now we have

$$3a - \sum_{i=1}^r b_i = 2, \quad a^2 - \sum_{i=1}^r b_i^2 = 0, \quad r < 9.$$

If $r < 7$, then by adding $b_8 = b_9 = 1$, we have

$$3a - \sum_{i=1}^9 b_i = 0, \quad a^2 - \sum_{i=1}^9 b_i^2 = -2, \quad b_8 = b_9 = 1.$$

This is exactly the same case as we have done for $C^2 < 0$. These are the cases (1)-(5) in our list.

Now we assume $r = 8$ in the following. Without loss, we could assume $b_1 \geq \cdots \geq b_8 \geq 1$. We have a quick estimate $8b_8^2 \leq a^2 \leq 121$, thus $b_8 \leq 3$. If $b_8 = 1$, we change it to $b_8 = 2$ and add $b_9 = 1$. Hence we have

$$3a - \sum_{i=1}^9 b_i = 0, \quad a^2 - \sum_{i=1}^9 b_i^2 = -4, \quad b_8 = 2, \quad b_9 = 1.$$

Rewrite it, we have

$$3a - \sum_{i=1}^9 b_i = 0, \quad \sum_{i=1}^9 (a - 3b_i)^2 = 36, \quad b_8 = 2, \quad b_9 = 1.$$

In total there are seven essentially different representations of 36 as a sum of 9 squares which are in the same residue class mod 3 and sum to 0:

$$\begin{aligned} 36 &= 3^2 + 3^2 + (-3)^2 + (-3)^2 + 0^2 + \cdots + 0^2 \\ &= (\pm 4)^2 + (\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2 + (\mp 2)^2 + (\mp 2)^2 + (\mp 2)^2 + (\mp 2)^2 \\ &= (\pm 5)^2 + (\pm 2)^2 + (\mp 1)^2 + (\mp 1)^2 + (\mp 1)^2 + (\mp 1)^2 + (\mp 1)^2 + (\mp 1)^2 + (\mp 1)^2 \\ &= (\pm 6)^2 + 0^2 + \cdots + 0^2. \end{aligned}$$

Up to the order of b_i , the solutions $(a; b_1, \dots, b_9)$ are

$$\begin{aligned} &(3b; b+1, b+1, b-1, b-1, b, \dots, b); \\ &(3b \pm 1; b \mp 1, b, b, b, b \pm 1, b \pm 1, b \pm 1, b \pm 1); \\ &(3b \pm 2; b \mp 1, b, b \pm 1, \dots, b \pm 1); \\ &(3b; b \pm 2, b, \dots, b). \end{aligned}$$

Note $b_8 = 2, b_9 = 1$, and b_8 has to change back to 1, this gives the list (6)-(10) in our statement. In particular, we remark that the last case in the above is impossible.

If $b_8 = 2$, we change it to $b_8 = 3$ and add $b_9 = 1$. Hence we have

$$3a - \sum_{i=1}^9 b_i = 0, \quad a^2 - \sum_{i=1}^9 b_i^2 = -6, \quad b_8 = 3, \quad b_9 = 1.$$

Rewrite it, we have

$$3a - \sum_{i=1}^9 b_i = 0, \quad \sum_{i=1}^9 (a - 3b_i)^2 = 54, \quad b_8 = 3, \quad b_9 = 1.$$

In total there are seven essentially different representations of 36 as a sum of 9 squares which are in the same residue class mod 3 and sum to 0:

$$\begin{aligned} 54 &= 3^2 + 3^2 + 3^2 + (-3)^2 + (-3)^2 + (-3)^2 + 0^2 + 0^2 + 0^2 \\ &= (\pm 5)^2 + (\pm 2)^2 + (\pm 2)^2 + (\mp 1)^2 + (\mp 1)^2 + (\mp 1)^2 + (\mp 1)^2 + (\mp 1)^2 + (\mp 4)^2 \\ &= (\pm 4)^2 + (\pm 4)^2 + (\pm 1)^2 + (\pm 1)^2 + (\mp 2)^2 + \cdots + (\mp 2)^2 \\ &= (\pm 6)^2 + (\mp 3)^2 + (\mp 3)^2 + 0^2 \cdots + 0^2. \end{aligned}$$

Up to the order of b_i , the solutions $(a; b_1, \dots, b_9)$ are

$$\begin{aligned} & (3b; b+1, b+1, b+1, b-1, b-1, b-1, b, b, b); \\ & (3b \pm 2; b \mp 1, b, b, b \pm 1, \dots, b \pm 1, b \pm 2); \\ & (3b \pm 1; b \mp 1, b \mp 1, b, b, b \pm 1, \dots, b \pm 1); \\ & (3b; b \pm 2, b \mp 1, b \mp 1, b, \dots, b). \end{aligned}$$

Note $b_8 = 3, b_9 = 1$, especially there are exactly one 1 among all b_i , we know the first and the third in the above are impossible. After changing b_8 back to 2, we have the list (11)-(13) in our statement.

Finally, if $b_8 = 3$, we change it to $b_8 = 4$ and add $b_9 = 1$. Hence we have

$$3a - \sum_{i=1}^9 b_i = 0, \quad a^2 - \sum_{i=1}^9 b_i^2 = -8, \quad b_8 = 4, \quad b_9 = 1.$$

Rewrite it, we have

$$3a - \sum_{i=1}^9 b_i = 0, \quad \sum_{i=1}^9 (a - 3b_i)^2 = 72, \quad b_8 = 4, \quad b_9 = 1.$$

In total there are seven essentially different representations of 36 as a sum of 9 squares which are in the same residue class mod 3 and sum to 0:

$$\begin{aligned} 72 &= 3^2 + 3^2 + 3^2 + 3^2 + (-3)^2 + (-3)^2 + (-3)^2 + (-3)^2 + 0^2 \\ &= (\pm 8)^2 + (\mp 1)^2 + \dots + (\mp 1)^2 \\ &= (\pm 7)^2 + (\pm 1)^2 + (\pm 1)^2 + (\pm 1)^2 + (\mp 2)^2 + \dots + (\mp 2)^2 \\ &= (\pm 6)^2 + (\pm 3)^2 + (\mp 3)^2 + (\mp 3)^2 + (\mp 3)^2 + 0^2 \dots + 0^2 \\ &= 6^2 + (-6)^2 + 0^2 \dots + 0^2 \\ &= (\pm 5)^2 + (\pm 2)^2 + (\pm 2)^2 + (\pm 2)^2 + (\mp 1)^2 + (\mp 1)^2 + (\mp 1)^2 + (\mp 4)^2 + (\mp 4)^2 \\ &= (\pm 4)^2 + (\pm 4)^2 + (\pm 4)^2 + (\mp 2)^2 + \dots + (\mp 2)^2. \end{aligned}$$

Up to the order of b_i , the solutions $(a; b_1, \dots, b_9)$ are

$$\begin{aligned} & (3b; b+1, b+1, b+1, b+1, b-1, b-1, b-1, b-1, b), \quad (3b \pm 2; b \mp 2, b \pm 1, \dots, b \pm 1); \\ & (3b \pm 1; b \mp 2, b, b, b, b \pm 1, \dots, b \pm 1), \quad (3b; b \pm 2, b \pm 1, b \mp 1, b \mp 1, b \mp 1, b, b, b, b); \\ & (3b; b+2, b-2, b, \dots, b), \quad (3b \pm 2; b \mp 1, b, b, b, b \pm 1, b \pm 1, b \pm 1, b \pm 2, b \pm 2); \\ & (3b \pm 1; b \mp 1, b \mp 1, b \mp 1, b \pm 1, \dots, b \pm 1). \end{aligned}$$

Note $b_8 = 4, b_9 = 1$, especially there are exactly one 1 and no 2 among all b_i , we know that only the second and the third in the above are possible. After switching b_8 back to 3, we have the list (14)-(15) in our statement. \square

After completing this paper, the author was kindly informed by Dusa McDuff and Felix Schlenk that they [24] have also obtained precisely this list of classes in Proposition 4.5 as corresponding to all full symplectic packings of a 4-ball by no more than 8 balls. In fact, our Propositions 4.5 and 3.2 altogether give an alternative argument of their full packing result. Precisely, the list of full packing corresponds to classes in $\partial\overline{\mathcal{P}} \cap \mathcal{C}_{M,K}$, which are the classes of K -symplectic spheres with self-intersection 0 (the list of Proposition 4.5) by Proposition 3.2.

The next lemma basically shows that the phenomenon discovered in [5] for (elliptic) ruled surfaces cannot happen for small rational surfaces.

Lemma 4.6. *Let $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$. Then*

- (1) *there is no irreducible curve C such that $C^2, K \cdot [C] \geq 0$ when $k < 9$.*
- (2) *For $k = 9$, the only such curve classes are $-mK = 3mH - mE_1 - \cdots - mE_9$, $m > 0$.*

Proof. Let $[C] = aH - b_1E_1 - \cdots - b_kE_k$ with $k \leq 9$. The conditions then read as

$$a^2 \geq b_1^2 + \cdots + b_k^2, \quad b_1 + \cdots + b_k \geq 3a.$$

Since it is a J -holomorphic curve class for a tamed J , with $C^2 \geq 0$, we have $a > 0$ by the Lemma 2.2. Hence

$$ka^2 \geq k(b_1^2 + \cdots + b_k^2) \geq (b_1 + \cdots + b_k)^2 \geq 9a^2.$$

It is a contradiction if $k < 9$. When $k = 9$ the equality holds if and only if $a = 3m > 0$ and $b_i = m$. \square

Especially, it shows that there are no curve classes such that $\dim_{SW}([C]) = 0$ and $g_J([C]) > 0$ when $k < 9$. There is a technical lemma for higher genus curve classes, which we will need later.

Lemma 4.7. *Let $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$, $k < 9$. Let $C = aH - b_1E_1 - \cdots - b_kE_k$ be a curve class.*

- (1) *If $C^2 \geq 0$, then only classes with $a \leq 2$ have $g_J(C) = 0$.*
- (2) *If $g_J(C) = 1$, then $C^2 \geq 9 - k$. The equality holds if and only if $C = 3H - E_1 - \cdots - E_k$.*

Proof. First $a > 0$ as in last lemma. Let $C^2 = c$, $g = g_J(C)$. Then by adjunction formula, the inequality $(b_1 + \cdots + b_k)^2 \leq k(b_1^2 + \cdots + b_k^2)$ reads as

$$(3a + 2g - 2 - c)^2 \leq k(a^2 - c).$$

When $a \leq 2$, the inequality holds only if $g = 0$ or $a = c = 2, g = 1, k = 8$. But the latter case would imply $b_i = \frac{1}{2}$ which is a contradiction.

By Proposition 4.2, $C^2 \geq 0$ if $g \geq 1$. Now we assume $g = 1$ and $a \geq 3$. Then the inequality reads as

$$(c - 9)(c - (9 - k)) + (3 - a)(6c - (3 + a)(9 - k)) \leq 0.$$

Hence if $c \leq 9 - k$, both terms are non-negative, with equality holds if and only if $a = 3$ and $c = 9 - k$. This in turn implies $C^2 = 9 - k$ if and only if $C = 3H - E_1 - \cdots - E_k$. \square

Finally we are ready to prove Theorem 1.2, which is reformulated in the following.

Theorem 4.8. *For rational 4-manifolds $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ with $k < 8$, the set of all the possible configurations of negative self-intersection curves for tamed almost complex structures are the same as the set for complex structures.*

Proof. We first show it for $k \leq 6$. Notice in this situation, by the above classification of irreducible negative curves, we know all the negative curves are rational curves and the intersections of them with a (different) -1 rational curve are either 0 or 1. To see this, we notice that any -1 rational curve class E has non-trivial Seiberg-Witten invariant, which allows us to apply Lemma 2.2 to conclude that $E \cdot H \geq 0$. Hence -1 curve classes are either E_i or (1) and (2) in the list of Proposition 4.4 when $k \leq 6$. Then Lemma 4.1 and Proposition 4.4 gives the result.

For our main claim, we use induction. The claim for $k = 1$ (and $S^2 \times S^2$) is proved in [19] and $k = 2$ is proved in Section 3. We look at configurations of negative curves satisfying the following three properties:

- all negative curves classes are chosen from the above classification and any two distinct ones intersect non-negatively;
- there is a rational cohomology class $a \in H^2(M, \mathbb{Q})$ such that $a \cdot a > 0$ and it pairs positively with these negative curve classes and classes with non-trivial Seiberg-Witten;
- any -1 rational curve classes are positive linear combinations of these negative curve classes and classes with non-trivial Seiberg-Witten.

In particular, the negative curve configuration for a tamed almost complex structure on $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$, $2 \leq k < 9$, satisfies these properties as we have shown before, in particular, Proposition 4.3. We will show that all the curve configurations satisfying above three properties could be realized as the negative curve configuration of a complex structure, at least when $k < 8$. We use induction and start it with $k = 2$ which is known to be true in section 3.

The idea of our proof is to proceed by starting with a negative curve configuration on $\mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ satisfying above three properties, combinatorially blowing it down to get a negative curve configuration on $\mathbb{C}P^2 \# (k - 1) \overline{\mathbb{C}P^2}$, proving it also satisfies the three properties and using the inductive hypothesis to realize this blown down configuration as a complex curve configuration, and then doing a complex blowup of this new configuration to realize the original curve configuration.

For this goal, our proof contains two steps. The first step is to prove that combinatorially blowing down a -1 curve from a negative curve configuration with the three properties will give a configuration of negative curves with the same three properties. Then by our induction assumption, this is a complex curve configuration. The second step is to show that we can apply a complex blowup of this new configuration to realize the original curve configuration.

We start now with the first step. Notice our curve configuration is not related to an almost complex structure at this moment, but we fix a canonical class $K = -3H + E_1 + \cdots + E_k$. For any $3 \leq k < 9$, any configuration of negative curves with the two properties includes at least a class of -1 rational curve by the same argument as Corollary 2.10 since it only involves adjunction formula. Let it be E_k . We combinatorially blow down this -1 curve, which means that remove E_k , and change any other curve classes to $C' = C + (C \cdot E_k)E_k$. Then remove all the non-negative curves from the configuration and keep the rest in the configuration if they still have negative square. We also define the new canonical divisor be $K' = K - E_k$. We check that

$$C'^2 = C^2 + (C \cdot E_k)^2, \quad K' \cdot C' = K \cdot C - C \cdot E_k.$$

Hence if we let $g_K(e) = \frac{1}{2}(e \cdot e + K \cdot e) + 1$, we have

$$g_{K'}(C') \geq g_K(C), \quad C'^2 - K' \cdot C' = C^2 - K \cdot C + (C \cdot E_k)^2 + C \cdot E_k \geq C^2 - K \cdot C.$$

Especially, all the curve classes C' in the new configuration still have $g_{K'}(C') \geq 0$. The equality holds if and only if $C \cdot E_k = 0$ or 1 . Especially, when $k \leq 6$, we have $g_{K'}(C') = 0$. Thus all the new negative classes C' are those classes in the previous classification since we only used adjunction formula and $g_{K'}(C') \geq 0$.

We claim that the new configuration still has a -1 rational curve class with respect to K' :

$$C'^2 = -1, \quad K' \cdot C' = K \cdot C' = -1.$$

To prove the claim, we notice that every class E in $\mathcal{E}_{E_k^\perp} \subset \mathcal{E}_K$, *i.e.* those classes in \mathcal{E}_K whose pairing with E_k is zero, is J -effective, *i.e.* $E = \sum a_i C_i$. Since $E \cdot E_k = 0$, we have $\sum a_i (C_i \cdot E_k) = 0$. Hence

$$E = \sum a_i C_i = \sum a_i C'_i = E'.$$

Since $K' \cdot E' = -1 < 0$, there is a C'_i with $K' \cdot C'_i < 0$. If $C_i'^2 < 0$, then we are done. If not, understand C'_i as a class in original $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$, which is orthogonal to E_k . We could continue this process for C'_i since it is not extremal and $K \cdot C'_i = -1$. This process will stop in finite steps because there is a positive lower bound of the pairing of a in our second properties with curve classes since it is a rational cohomology class. Finally we will get a -1 rational curve class. This curve class is orthogonal to E_k since $C' \cdot E_k = 0$.

This is a part of our induction assumption. It is also direct to check that different new negative curve classes intersect non-negatively. On the other hand, to show all classes C' with $SW(C') \neq 0$ and $E_k \cdot C' = 0$ are positive linear combinations of these new negative curves and other Seiberg-Witten nontrivial classes, we only need to show it for $C' \in \mathcal{E}_{E_k^\perp}$ as argued in the cone theorem. We can always write $E \in \mathcal{E}_{E_k^\perp}$ as linear combination of C_i 's in the original configuration and those with non-trivial Seiberg-Witten invariant. Since $C' \cdot E_k = 0$, C' is the linear combination of C'_i with the same coefficients where $C'_i = C_i + (C \cdot E_k)E_k$. It is direct to see that if $SW(C_i) \neq 0$ then $SW(C'_i) \neq 0$ as well. Now we assume C_i is originally a negative curve. If C'_i is a negative curve, we are fine. If C'_i is no longer a negative curve, from the classification of negative curves earlier in this section, we see that $SW(C'_i) \neq 0$. This finishes the first step.

Then we prove the second step: to show that we can reverse the process using complex blowups, at least when $k < 8$. We first discuss the case when $k \leq 6$. By induction, the above new configuration is realized by a complex structure on $\mathbb{C}P^2 \# (k-1)\overline{\mathbb{C}P^2}$. Then we do complex blow up as following. When we remove E_k , if self-intersections and mutual intersections are unchanged for new negative curves configuration comparing to the the corresponding part of the original one, then we blow up a generic point of $\mathbb{C}P^2 \# (k-1)\overline{\mathbb{C}P^2}$. Remember all the negative curves are rational curves and the intersections of them with a (different) -1 rational curve are either 0 or 1, so we only have the following three cases. If it changes self-intersection of certain curve but not the mutual intersections, we blow up a generic point of this curve. If new intersection is introduced, we blow up this intersection. The three different situations correspond to involving zero, one or more than one negative curves respectively when we do the combinatorial blow down. We want to make sure such “generic” choices exist. The first case happens if some -1 curves become square 0 classes when blowing down E_k . We check that these new square 0 classes are nef with respect to any possible tamed almost complex structures. This is apparent since the new class is $C = C_1 + C_2$ where C_i are -1 curves (one of them is E_k) and $C_1 \cdot C_2 = 1$. Hence by Proposition 4.5 in [19], for any given point of $\mathbb{C}P^2 \# (k-1)\overline{\mathbb{C}P^2}$, we have a possibly reducible rational curve in this class. Moreover, there is a unique such curve passing through any given point. By Theorem 1.4 of [20], reducible curves happen only when all components are in the new negative curve configuration. Hence, for any points on $\mathbb{C}P^2 \# (k-1)\overline{\mathbb{C}P^2}$ outside the negative curve locus, we have a smooth curve in the class C . Hence a blow up will send this curve to a negative curve. Also, by the classification above, no other types of negative curves are possible to be produced. The second and third cases are similar to argue. In these cases, if the component(s) we want to blow up are part of the reducible curve in a square 0 rational curve class C' . Then the -1 class $C' - E$ in the blow up is represented by a reducible curve. When the point blown up is a generic point on a negative

curve, this is the second case. If it is an intersection point, this is the third case above. Otherwise, we will have a smooth rational curve as above. We thus finish the proof for $k \leq 6$.

The case of $k = 7$ introduces a new type of -1 curves, *i.e.* class (3) in Proposition 4.4. Thus there are possibly a negative curve and a -1 rational curve having intersection number 2. All the argument for $k \leq 6$ went through, except now we will have a square 3 class $C' = 3H - E_1 - \cdots - E_6$ (the corresponding $C = 3H - E_1 - \cdots - E_6 - 2E_7$) to deal with. This is a class of J -genus 1. As above, by induction, we have a complex structure on $M' = \mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$ such that the combinatorial blown down of the negative curve configuration is realized. We look at the the class C' . When we blow up at M' at one point, we will have an embedded rational curve in class E_7 , and a possibly reducible curve in class C . Hence for any given point on M' there is a non-embedded (possibly reducible) subvariety in class C' whose one singular (nodal, cusp or multiple) point is that point. These three types of singularities correspond to E_7 is secant to, tangent to, or an irreducible component of the subvariety representing C .

Since the class C' is J -nef, we know there are two possible types of reducible curves: there is an elliptic curve class as a component, or all the curve classes are rational. If the first case happens, then $[C'] = m_0[C_0] + \sum m_i[C_i]$ where $g_J(C_0) = 1$ and $g_J(C_i) = 0$ for other i . Since C' is J -nef, $[C'] \cdot [C_i] \geq 0$. By Lemma 4.7, we know $[C']^2 \leq [C_0]^2 = -K_{M'} \cdot [C_0] = [C'] \cdot [C_i]$. The inequality holds if and only if $C' = C_0$. Hence in this case the curve in class C' is irreducible.

For the second case, let $C' = \sum m_i e_i$. Remember the class C' is nef. Use $1, \dots, l$ to label the curves whose class has negative self-intersection. Then

$$C' \cdot C' = \sum_{j=l+1}^n (m_j^2 e_j \cdot e_j + m_j e_j \cdot (C' - m_j e_j)) + \sum_{i=1}^l m_i e_i \cdot C'.$$

All terms are non-negative. Since the corresponding subvariety is connected by Proposition 4.25 of [20], the second term is positive. We want to show that for reducible curves, the double points are on some negative component. Hence we could assume $n - l > 1$ otherwise we are done since double points are on some negative curves. Without loss, we assume $m_i = 1$ for $i > l$ as in [19].

First it is impossible to have $l = 0$ and $n \geq 2$. If so there will be a cycle in the graph of the corresponding subvariety. Hence the second term will contribute at least 4. If $l > 0$ and $n - l > 2$, then it is possible only when $n - l = 3$ and $e_i \cdot e_j = 0$ for $i, j > l$. Hence, for this case, all double points are on some negative curves. Now we are left with the case of $n - l = 2$. Then the intersection of the two curves cannot be greater than 1. Otherwise the second term contributes at least 4. It also cannot be 1 since there are at least two more intersections with negative curve components. To see this, let e_i be a class with $i > l$. If it does not intersect the negative curves, then

$e_i \cdot (C' - e_i) = 1$. Hence it is straightforward to calculate that $g_J(C' - e_i) = 1$ and $(C' - e_i)^2 = 2 - C' \cdot e_i = 1 - e_i^2 \leq 1$. This contradicts to Lemma 4.7 (2). Since $n - l = 2$, we know the second term is at least 4. If the intersection is 0, then by the light lemma, they are cohomologous. This implies that except possibly one or two cohomologous square 0 class, all the other connected components of the reducible curve are negative curves. Hence the non-generic blow-ups happen when the point is on the negative curve locus of the reducible curve. Thus the conclusion follows from the same argument for the case of $k \leq 6$. \square

When $k = 8$, we have three more classes: (4)-(6) in Proposition 4.4. Hence there are negative curves with mutual intersection 3. A similar argument should be enough to give a proof. The only difference is that a “generic” blowup is no longer blowing up outside the negative locus. For example,

$$6H - 3E_1 - 2E_2 - \cdots - 2E_8 = (3H - 2E_1 - E_2 - \cdots - E_8) + (3H - E_1 - \cdots - E_8).$$

Before blowing up E_1 , both curves have positive squares (one nodal curve and one smooth curve intersect at the node). However, it is a non-generic phenomenon since it is a reducible curve. Another interesting new feature of this reducible curve is one of its component is of genus one although the original class is of genus 0. Recall this cannot happen if the original class is J -nef by [20]. However, the following question still makes sense.

Question 4.9. *Suppose M is not diffeomorphic to $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$. Let $E \in \mathcal{E}_{K,J}$. Is it true that for any subvarieties $\Theta = \{(C_i, m_i)\}$ in class E , i.e. $E = \sum m_i [C_i]$, we have $g_J([C_i]) = 0$?*

For interested readers, the above decomposition of an exceptional class could also be seen from blowing down certain elliptic fibration of $E(1) \cong \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$. More precisely, fix an elliptic fibration of $E(1)$ with an I_2 fiber and a -1 section E (the existence of such fibration should be well known, see e.g. [2]). Then the I_2 is constituted by two -2 rational curves C_1 and C_2 . Choose a generic fiber F . We know $[C_1] + [C_2] = [F] = -K$. Let the section E intersect with C_2 (and F). After blowing it down, C_1 is unchanged and F becomes a smooth elliptic curve F' with self-intersection 1. We see C_1 and F' are disjoint, and $[C_1] + [F'] \in \mathcal{E}_K$ by the adjunction formula.

Theorem 1.1 in [1] states that the inclusion of the space of compatible integrable complex structures into the space of all compatible almost complex structures is a weak homotopy equivalence for a rational ruled surface. Our Theorem 4.8 indicates that it may hold for $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with $k < 9$.

4.2. Complex Configurations for small rational surfaces. From Theorem 4.8, to know all the possible curve configurations for tamed almost complex structures, we only need to know that for complex structures when our underlying manifold is a small rational surface. This subsection summarizes all such possibilities for rational surfaces of Euler number no greater

than 6. Based on the all possible curve configurations for $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$, we will discuss the limitation of the construction of almost Kähler forms using spherical classes as in [19].

For $S^2 \times S^2$, the possible types are Hirzebruch surfaces \mathbb{F}_{2n} . So the only irreducible negative curve is a $-2n$ curve which is in class $A - nB$ (or $B - nA$) where $A = [\{pt\} \times S^2]$, $B = [S^2 \times \{pt\}]$.

For $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, the possible types are \mathbb{F}_{2n+1} . So the only negative curve is in class $(n+1)E - nH$.

For $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$, we view this as blow up of \mathbb{F}_{2n+1} . We can either blow up at a point on the unique negative curve of \mathbb{F}_{2n+1} , or blow up at a point not on it. For the first case, our configuration is E_2 , $H - E_1 - E_2$ and $(n+1)E_1 - nH - E_2$. For the latter case, our negative curves are E_2 , $H - E_1 - E_2$ and $(n+1)E_1 - nH$.

In all the above cases, the dual of the curve cone is the J -spherical cones S_J which is the Kähler cone. As we see in Theorem 3.10 and in [19], these configurations realize all the possible configurations of negative curves for any almost complex structures on $S^2 \times S^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$.

For $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$, it is a further blowup at certain points on some complex structure of $\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2}$. we can blow up

- at a point not on negative curves (a generic point), then the negative curves are E_3 , E_2 , $H - E_1 - E_2$, $H - E_1 - E_3$ and $(n+1)E_1 - nH - E_2$, or E_3 , E_2 , $H - E_1 - E_2$, $H - E_1 - E_3$, $H - E_2 - E_3$ (if $n = 0$) and $(n+1)E_1 - nH$;
- at a generic point of E_2 , then the curves are E_3 , $E_2 - E_3$, $H - E_1 - E_2$ and $(n+1)E_1 - nH - E_2$, or E_3 , $E_2 - E_3$, $H - E_1 - E_2$ and $(n+1)E_1 - nH$;
- at a generic point of $H - E_1 - E_2$, then the curves are E_3 , E_2 , $H - E_1 - E_2 - E_3$ and $(n+1)E_1 - nH - E_2$, or E_3 , E_2 , $H - E_1 - E_2 - E_3$ and $(n+1)E_1 - nH$;
- at a generic point of $(n+1)E_1 - nH - E_2$ or $(n+1)E_1 - nH$, we get E_3 , E_2 , $H - E_1 - E_2$, $H - E_1 - E_3$ and $(n+1)E_1 - nH - E_2 - E_3$, or E_3 , E_2 , $H - E_1 - E_2$, $H - E_1 - E_3$ and $(n+1)E_1 - nH - E_3$;
- at the intersection point of $H - E_1 - E_2$ and $(n+1)E_1 - nH$, then the curves are E_3 , E_2 , $H - E_1 - E_2 - E_3$ and $(n+1)E_1 - nH - E_3$;
- at the intersection point of $(n+1)E_1 - nH - E_2$ and E_2 , then the curves are E_3 , $E_2 - E_3$, $H - E_1 - E_2$ and $(n+1)E_1 - nH - E_2 - E_3$;
- at the intersection point of E_2 and $H - E_1 - E_2$, then the curves are only E_3 , $E_2 - E_3$, $H - E_1 - E_2 - E_3$ and $(n+1)E_1 - nH - E_2$, or E_3 , $E_2 - E_3$, $H - E_1 - E_2 - E_3$ and $(n+1)E_1 - nH$;

Notice in the last case, we only have one irreducible -1 rational curve, which is in class E_3 . For all the others, we have at least two smooth -1 rational curves. Then we can show that if for an almost Kähler structure

the configuration of the negative curves is like the first six cases, the Nakai-Moishezon type theorem as Theorem 3.12 holds since J -spherical cones are equal to the Kähler cones.

Proposition 4.10. *If the configuration of negative curves for an almost Kähler structure is one of the first six bullets listed above, we have*

$$\mathcal{K}_J^c = \mathcal{S}_J = \mathcal{P}_J = A_J^{\vee, >0}(M).$$

Proof. First notice $\mathcal{P}_J = A_J^{\vee, >0}(M)$. This is because, for any J , $A_J^{\vee, >0}(M)$ is contained in polytope with vertices H , $H - E_1$, $H - E_2$, $H - E_3$ and $2H - E_1 - E_2 - E_3$.

Then notice $\mathcal{S}_J \subset \mathcal{K}_J^c$ and $\mathcal{K}_J^c \subset \mathcal{P}_J$. Thus we could reduce the rest to show that $\mathcal{S}_J = \mathcal{P}_J$. To prove $\mathcal{S}_J = \mathcal{P}_J = A_J^{\vee, >0}(M)$, we notice for the first case $A_J^{\vee, >0}(M)$ is another triangular bipyramid with all vertices are spherical classes. These vertices could be represented or approximated by classes in \mathcal{S}_J . For the rest, they are all tetrahedra with at least two faces (thus span the tetrahedron) determined by -1 classes which can be generated by spherical classes. \square

If the negative curves are E_3 , $E_2 - E_3$, $H - E_1 - E_2 - E_3$ and $(n + 1)E_1 - nH - E_2$, as in the last case, then the corners of $A_J^{\vee, >0}(M)$ are $(2n + 3)H - (2n + 1)E_1 - E_2 - E_3$, $(n + 2)H - (n + 1)E_1 - E_2$, $(n + 1)H - nE_1$ and $H - E_1$. The first one cannot be represented or approximated by a sphere, while the other three classes are all spheres. All the spherical classes are on the boundary (or more precisely, on the edges) of the triangular bipyramid with vertices H , $H - E_1$, $H - E_2$, $H - E_3$ and $2H - E_1 - E_2 - E_3$. While the class $(2n + 3)H - (2n + 1)E_1 - E_2 - E_3$ is in the interior of the hexahedron.

If the curves are E_3 , $E_2 - E_3$, $H - E_1 - E_2 - E_3$ and $(n + 1)E_1 - nH$, then the corners of $A_J^{\vee, >0}(M)$ are $(2n + 2)H - 2nE_1 - E_2$ and other spherical classes.

In other words, in both subcases of Case 7, our spherical classes only span a face of the dual of curve cone. This is point the techniques in [30, 19] does not work. However, this case will be covered by the inflation method in section 5, see Theorem 1.6.

4.3. Configurations of smooth -1 rational curves. In addition to study the possible configurations of all smooth negative curves, we could also look at the configurations of smooth -1 rational curves.

In this section, we assume $M_k = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with $k \geq 1$. Dusa McDuff asks a couple of questions on possible numbers of -1 rational curves.

Question 4.11 (McDuff). (1) *What are the possible maximal numbers l of disjoint embedded -1 rational curves for tamed almost complex structures on M_k ?*

(2) *What are the possible numbers of embedded -1 rational curves for tamed almost complex structures on M_k , especially when $k \geq 9$?*

We first give a few remarks on Question 4.11 (2). It is very direct to work out the possible numbers of embedded -1 rational curves for a complex structure on M_k with $k < 9$. For examples, from section 4.2, we know there could be 0 or 1 embedded -1 rational curves on M_1 , 2 or 3 embedded -1 rational curves on M_2 , 1, 2, 3, 4, or 6 embedded -1 rational curves on M_3 . Theorem 1.2 implies that at least when $k < 8$, working with a tamed almost complex structure would not produce more possibilities.

Now, let us work with the first part of Question 4.11 (1). Apparently, $l \leq b^-(M_k) = k$. By Corollary 2.10, we have $l \geq 1$ when $k \geq 2$. Our Theorem 3.6 says that when $k = 2$, there are at least two -1 rational curves. However, from the discussion in section 4.2, the possible value of l could be 1 or 2. The following result implies for a general $k \geq 1$, any integer l with $1 \leq l \leq k$ could be realized. This is based on an argument delivered to the author by McDuff.

Proposition 4.12. *Let $M_k = \mathbb{C}P^2 \# k \overline{\mathbb{C}P^2}$ with $k \geq 3$ and l an integer with $1 \leq l \leq k$. There is an integrable (Kähler) J on M_k with exactly l embedded -1 rational curves. Moreover, these rational curves are disjoint.*

Proof. We start with a line on $\mathbb{C}P^2$. We blow up at l distinct points on this line, call these exceptional curves E_1, \dots, E_l . We then blow up at the intersection of E_l with the line. Call the new exceptional curve E_{l+1} , and blow up its intersection with the line. Continue this process to do $k - l$ blow ups. Now we have a complex structure on M_k which contains negative curves in classes

$$H - E_1 - \dots - E_k, E_1, \dots, E_{l-1}, E_l - E_{l+1}, \dots, E_{k-1} - E_k, E_k.$$

If there is another embedded -1 rational curve, say E , in class $dH - \sum m_i E_i$, then we have

$$d \geq m_1 + \dots + m_k, \text{ and } m_i \geq 0, \quad \forall i.$$

In particular, it implies $d \geq 0$. Moreover $d \neq 0$ otherwise all m_i have to be 0 and $E = 0$.

Recall that a class $x_0 H - \sum x_i E_i$ is called *ordered* if $x_1 \geq \dots \geq x_k$. An ordered vector is *reduced* if $x_0 \geq x_1 + x_2 + x_3$ and $x_i \geq 0$ for all i . Hence the class of E , when ordered, is a reduced class. However, by Lemma 3.4 of [13], there is no reduced class in \mathcal{E}_K . Hence E_1, \dots, E_{l-1}, E_k are the only exceptional curves, which are disjoint to each others. \square

We notice this result also gives a partial answer for Question 4.11 (2): when $k \geq 3$, there could be 1 to k embedded -1 rational curves.

There is a different construction for $l = 1$ which is due to McDuff. By section 3.3, we have an integrable complex structure J_2 on M_2 with negative curves

$$(n+1)E_1 - nH - E_2, \quad H - E_1 - E_2, \quad E_2.$$

Then blow up the intersection point of $H - E_1 - E_2$ and E_2 to get J_3 on M_3 . Then we blow up inductively the intersection point of $E_{i-1} - E_i$ and

E_i for $3 \leq i \leq k$ to get J_k on X_k with negative curves

$$(n+1)E_1 - nH - E_2, \quad H - E_1 - E_2 - E_3, \quad E_i - E_{i+1}, i = 2, \dots, k-1, \quad E_k.$$

Use the similar argument as Proposition 4.12, we can show that the only -1 rational curve lies in the class E_k . We leave the full detail to the interested readers.

Since we can choose any $n \geq 0$, there are infinitely many possible configurations for $l = 1$ in Proposition 4.12.

Proposition 4.12 could be extended to a general symplectic 4-manifold.

Theorem 4.13. *Let M be diffeomorphic to $N \# k\overline{\mathbb{C}P^2}$ with $k \geq 1$ and N a minimal symplectic 4-manifold. We assume M is not diffeomorphic to one point blow up of an S^2 bundle over surface. Given $1 \leq l \leq k$, there is a tamed J on M such that there are exactly l embedded -1 rational curves. Moreover, these rational curves are disjoint.*

Proof. When M is a rational surface, it follows from Proposition 4.12.

When M is irrationally ruled, we could also choose such a Kähler structure. We start with $N = S^2 \times \Sigma_h$ endowed with product complex structure. Denote the fiber class by T . Blow up at one point, we have two negative curves $E_1, T - E_1$. Further blow up $l - 1$ times at distinct points on $T - E_1$ other its intersection with E_1 . Then we further blow up at the intersection of E_l with the line. Call the new exceptional curve E_{l+1} , and blow up its intersection with the line. Continue this process to do $k - l$ blow ups. In total, we have done k blow ups, and $M = N \# k\overline{\mathbb{C}P^2}$. Now we have negative curves in classes

$$T - E_1 - \dots - E_k, E_1, \dots, E_{l-1}, E_l - E_{l+1}, \dots, E_{k-1} - E_k, E_k.$$

For irrational ruled surfaces, -1 rational curves could only appear in classes E_i and $T - E_i$ for $1 \leq i \leq k$. However, $(T - E_i) \cdot (T - E_1 - \dots - E_k) = -1 < 0$ for any $i \leq k$. Hence E_1, \dots, E_{l-1}, E_k are the only exceptional curves, which are disjoint to each others.

For a non-rational and non-ruled symplectic manifold, we could choose a tamed almost complex structure on N such that in a small ball, it is integrable. Then we first blow up l distinct points in this ball and then blow up consecutively on E_l . Again E_1, \dots, E_{l-1}, E_k are the only exceptional curves, which are disjoint to each others. \square

4.4. Irrational ruled surfaces. In this section, we discuss the cases of irrational ruled surfaces and prove Theorem 4.14.

In general, the complex structures of non-rational ruled surfaces are much more complicated than that of rational ones. Any such minimal surface M could be viewed as the projectivization $\mathbb{P}(E)$ of a vector bundle of dimension two over Σ_g . The curve cone behaves quite different when E is unstable from it is semi-stable. When E is unstable, *e.g.* $E = L \oplus \mathcal{O}$, the corresponding ruled surface $\mathbb{P}(E)$ has a negative curve. This is because by definition, we have a line bundle quotient A of negative degree a . Then $C = \mathbb{P}(A)$ is an

effective curve in the class $aT + U$ with $C^2 = 2a, 2a + 1 < 0$. Recall that T is the class of the fiber S^2 and U is the class of a section with $U^2 = 0$ or 1 . In this case, the curve cone $A(M)$ is always closed.

In contrast, when E is semi-stable, the curve cone has different features. For convenience, we assume E has even degree, and after twisting a line bundle we can then suppose $\deg E = 0$. First this is always true that the nef cone is the same as the closure of the curve cone which is the first quadrant of the U - T plane. This is because if there is an irreducible curve C in the class $aT + bU$, then $C \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m) \otimes \pi^*A) = \Gamma(S^m E \otimes A)$ for some integer $m \geq 0$ and some line bundle A . It would imply $a \geq 0$ by semi-stability. On the other hand, $b \geq 0$ since there is always an irreducible curve in class T and thus $[C] \cdot T \geq 0$. There is a famous example of Mumford showing that the curve cone might not be closed by the existence of the bundle E over Σ_g with $g > 1$ such that $\Gamma(S^m E \otimes A) = 0$ for all $m \geq 1$ whenever $\deg A \geq 0$.

The above discussion suggests that bizarre things may happen for non-negative curves. See the discussion in the end of this section. However, the configuration of negative curves is always very simple.

Theorem 4.14. *For minimal irrational ruled surfaces, i.e. S^2 bundles over $\Sigma_{h \geq 1}$, the set of all the possible configurations of negative self-intersection curves for tamed almost complex structures are the same as the set for complex structures.*

Proof. We divide our discussion in two cases.

- $\Sigma_h \times S^2$, $h \geq 1$

In this case, let U be the class of the base Σ_h and T be the class of the fiber S^2 . Then the canonical class $K = -2U + (2h - 2)T$. We suppose F is an irreducible J -holomorphic curve with negative square, and $[F] = aU + bT$ for some integers a and b . Then $a \cdot b < 0$.

The adjunction formula tells us that

$$-2b + (2h - 2)a + 2ab = 2g(F) - 2.$$

If we project F to the base Σ_h , the degree of the map is a . Since Σ_h has genus at least one, we have

$$2g(F) - 2 \geq a(2h - 2).$$

Hence we have

$$-2b + (2h - 2)a + 2ab \geq a(2h - 2),$$

and in turn,

$$2b(a - 1) \geq 0.$$

Since $a \cdot b < 0$, it implies $a = 1$ and $b < 0$. For the configuration, we know that at most one class of the type $U - kT$ with $k \geq 0$ could appear because the negative intersection of each other.

On the other hand, we could also show that $U - kT$ is the class of some complex curve for a complex structure on $\Sigma_h \times S^2$. Suppose L is a holomorphic line bundle with degree $2k \geq 0$. Then projectivization $\mathbb{P}(L \oplus \mathcal{O})$ is topologically $\Sigma_h \times S^2$. Moreover, the section $S_{-k} = \mathbb{P}(L \oplus 0)$ of the \mathbb{P}^1 bundle has self-intersection $-2k$, which is in the class $U - kT$.

- Non-trivial S^2 bundles over Σ_h , $h \geq 1$

Let U be the class of a section with square 1 and T be the class of the fiber. Then the canonical class $K = -2U + (2h - 1)T$. We suppose F is an irreducible J -holomorphic curve with negative square, and $[F] = aU + bT$ for some integers a and b . Then $a \cdot (a + 2b) < 0$.

The adjunction formula tells us that

$$-2b + (2h - 1)a - 2a + a^2 + 2ab = 2g(F) - 2 \geq a(2h - 2),$$

which is equivalent to say that

$$(a + 2b)(a - 1) \geq 0.$$

This again implies $a = 1$ and $b < 0$, which shows the negative curves are in classes $U - kT$.

The rest of the argument is exactly the same as the case of $\Sigma_h \times S^2$. Suppose L is We only a holomorphic line bundle L of degree $2k - 1$. The section $S_{-k} = \mathbb{P}(L \oplus 0)$ is in the class $U - kT$. \square

Remark 4.15. Notice that the Seiberg-Witten invariant calculation shows that there is a curve in class $aU + bT$, $a, b > 0$ (let us focus on the trivial bundle case here, similar for the nontrivial bundle case), if and only if $ab + b + a - ah \geq 0$. This implies the closure of curve cone is still the $U - T$ plane. However, it is intriguing to see whether there is a generic complex structure in the sense that only curve classes are the Seiberg-Witten non-trivial classes.

We now give an interpretation of the example in [5]. Consider the non-trivial S^2 bundle over T^2 . The classes U and T have the same meaning as above. Then the canonical class $K = -2U + T$. Consider the class $-2K$, it is the class of a square zero torus and its Seiberg-Witten dimension is 0. Hence, generically we only have a unique J -holomorphic curve in this class. The key observation of [5] is this is not true for complex structures: for any complex structures, there is always a J -holomorphic tori in class $-2K$ passing through any given point. Hence, after one blow-up at any point, we have a -1 J -holomorphic torus (possibly reducible) in class $4U - 2T - E$. Notice its Seiberg-Witten dimension is negative, so generically there is no curve in this class.

It is interesting to see that if we blow down along the other -1 curve, the one in class $T - E$, we will have $S^2 \times T^2$. The curve class is $4U + T$ in it (now our $U^2 = 0$), which is a genus 4 class. Since the previous class in one

point blow-up is represented as $4U + T - 3E$ in our new basis, it implies every point is a triple point of a holomorphic curve in class $4U + T$ for any complex structures, which is of course not a generic phenomenon.

5. NAKAI-MOISHEZON DUALITY FOR MANIFOLDS WITH ABUNDANT NEGATIVE SELF-INTERSECTION CURVES

In this section, we assume the tamed almost complex 4-manifold (M, J) has sufficiently many negative curves, such that \mathcal{P}_J has no round boundary. We say there is no round boundary if the boundary is a cone over a polytope. Thus any class e with $e^2 = 0$ should have $e \cdot C = 0$ for some $C \in A_J(M)$.

As mentioned in the introduction, besides the subvarieties-current-form strategy, there is another way to attack Question 1.4. This is our main focus in this section. Alongside the main theorem in [18], we will need to construct J -tamed symplectic forms from an existing one. We use three operations in this section. The first one is the J -tamed inflation along curves with negative self-intersection (and sometimes along curves of square 0), as described in Theorem 5.4 (Theorem 5.3 respectively). The second one is the summing of two J -tamed symplectic forms. The third one is rescaling, i.e. multiplying any J -tamed symplectic form with a positive number. The latter two make sense since \mathcal{K}_J^t is a convex cone.

Let us begin with several lemmas. First, let us determine the polytopic boundary of \mathcal{P}_J .

Lemma 5.1. *Let $h_J^+(M) = b^-(M) + 1$ and $\mathcal{P}_J \neq \emptyset$.*

- (1) *If $b^-(M) > 1$, all the boundary hyperplanes are determined by negative curves.*
- (2) *If $b^-(M) = 1$, all the boundary hyperplanes are determined by non-positive curves.*
- (3) *If $b^-(M) = 0$, then \mathcal{P}_J is a single ray.*

Proof. Recall $\mathcal{P}_J \subset H_J^+(M)$, The third item is self-evident. Now we can assume $b^-(M) > 0$.

By the light cone lemma, if A, B are classes in H_J^+ with $A^2 > 0, B^2 \geq 0$, then we have $A \cdot B > 0$. This implies any positive curves cannot determine a polytopic boundary of \mathcal{P}_J .

If A, B are classes in H_J^+ with $A^2, B^2 \geq 0$, then we have $A \cdot B \geq 0$. And the equality holds if and only if A is proportional to B . This implies any square 0 curve classes will contribute a ray in the polygonal boundary of \mathcal{P}_J . It is a boundary hyperplane only when the cone \mathcal{P}_J has dimension 2, i.e. $b^-(M) = 1$.

These give the proof of the first two facts. □

The next one is on the geometric property of a general \mathcal{P}_J .

Lemma 5.2. *Let C_i 's be the irreducible J -holomorphic curves in $A_J(M)$. If $C_i^2 < 0$, for any class $A \in \mathcal{P}_J$, and any $0 < \epsilon < \frac{A \cdot [C_i]}{-C_i^2}$, the class $(A + \epsilon[C_i]) \in \mathcal{P}_J$. If $C_i^2 \geq 0$, $A + \epsilon[C_i] \in \mathcal{P}_J$ for any $\epsilon > 0$.*

Proof. First, $(A + \epsilon[C_i]) \cdot [C_i] = A \cdot [C_i](1 + \epsilon \frac{C_i^2}{A \cdot [C_i]}) > 0$.

When $i \neq j$, $(A + \epsilon[C_i]) \cdot [C_j] = A \cdot [C_j] + \epsilon[C_i] \cdot [C_j] > 0$ because both terms are positive.

Finally, $(A + \epsilon[C_i])^2 = A^2 + \epsilon A \cdot [C_i] + (A + \epsilon[C_i]) \cdot [C_i] > 0$. \square

Among the 3 operations mentioned above for constructing J -tamed symplectic form, the J -tamed inflation is the most important one. One of the most effective tools to determine the symplectic cone of a 4-manifold is the (positive) symplectic inflation process introduced by Lalonde and McDuff in [11] along symplectic curves with non-negative self-intersection. In [17], this construction is extended to the case of negative self-intersection curves. There is also a corresponding J -tamed version of it. McDuff, in [22], proved the following result regarding the existence of (embedded) J -holomorphic curves with non-negative self-intersection.

Theorem 5.3 (McDuff). *Let J be a τ_0 -tame almost complex structure on a symplectic 4-manifold (M, τ_0) that admits an embedded J -holomorphic curve Z with $Z \cdot Z \geq 0$. Then there is a family τ_λ , $\lambda \geq 0$, of symplectic forms that all tame J and have cohomology class $[\tau_\lambda] = [\tau_0] + \lambda[Z]$.*

More recently, Buse (in [4]) provided the corresponding version when J -holomorphic curves with negative self-intersection are in presence.

Theorem 5.4 (Buse). *Fix a symplectic 4-manifold (M^4, J, τ_0) such that J is any τ_0 -tame almost complex structure. Assume that M admits an embedded J -holomorphic curve $u : (\Sigma, j) \rightarrow (M^4, J)$ in a homology class Z with $Z^2 = -m$. For all $\epsilon > 0$ there exist a family of symplectic forms τ_μ all tame J which satisfy*

$$[\tau_\mu] = [\tau_0] + \mu Z$$

for all $0 \leq \mu \leq \frac{\tau_0(Z)}{m} - \epsilon$.

For the convenience of discussion, let us introduce the notion of the *formal J -inflation*.

Definition 5.5. *An operation on a class A is called a formal J -inflation along the cohomology class $C \in H_J^+(M)$ with $A \cdot C \geq 0$ and $C^2 < 0$, if A is transformed to $A + \epsilon C$ with $0 < \epsilon \leq \frac{A \cdot C}{-C^2}$. When $\epsilon = \frac{A \cdot C}{-C^2}$, we call it a maximal formal J -inflation.*

A self-evident fact for this definition is that a class obtained from formal J -inflation could be approximated by genuine J -tamed symplectic inflations if the class $A \in \mathcal{K}_J^t$ and C is the class of an embedded J -holomorphic curve

with $C^2 < 0$. This fact will be used frequently in this section along with the main result in [18] that $\mathcal{K}_J^t \cap H_J^+(M) = \mathcal{K}_J^c$.

Lemma 5.2 demonstrates that the closure of the dual cone \mathcal{P}_J is closed under the operation of formal J -inflation. Because \mathcal{P}_J is a convex cone, it is also closed under summing and rescaling. Thus $\overline{\mathcal{P}_J}$ is closed under all the three operations. Moreover, after the three operations, the class will still stay in the same connected component of \mathcal{K}_J^t as beginning.

Lemma 5.6. *Suppose $h_J^+(M) = b^-(M) + 1$. Let C_1 and C_2 be two smooth J -holomorphic curves with negative intersection, which provide two hyperplane pieces \mathcal{C}_1 and \mathcal{C}_2 of the boundary respectively. If the intersection $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \overline{\mathcal{P}_J} \neq \emptyset$, then $([C_1] \cdot [C_2])^2 \leq C_1^2 \cdot C_2^2$. Moreover, the equality holds if and only if there is a unique ray in the above intersection which is spanned by $[C_1] - \frac{[C_1] \cdot [C_2]}{C_2^2} [C_2]$.*

Proof. Let us assume $[C_1] \cdot [C_2] \neq 0$ from now on. Suppose $([C_1] \cdot [C_2])^2 > C_1^2 \cdot C_2^2$ and there is a class $A \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \overline{\mathcal{P}_J}$.

Hence we can construct a class $[C_2] - \frac{C_1 \cdot C_2}{C_1^2} [C_1]$. Notice this class pairs non-negatively with all the curve classes. Moreover,

$$([C_2] - \frac{C_1 \cdot C_2}{C_1^2} [C_1])^2 = C_2^2 - \frac{C_1 \cdot C_2}{C_1^2} > 0.$$

Since $A \cdot ([C_2] - \frac{C_1 \cdot C_2}{C_1^2} [C_1]) = 0$ and $\mathcal{P}_J \subset H_J^+(M)$, we have $A^2 < 0$ by applying the light cone lemma to the $(1, b^-)$ space $H_J^+(M)$. This is a contradiction.

The equality case goes similarly, except for the last step we have A is proportional to $[C_2] - \frac{C_1 \cdot C_2}{C_1^2} [C_1]$. Notice $[C_2] - \frac{C_1 \cdot C_2}{C_1^2} [C_1]$ and $[C_1] - \frac{C_1 \cdot C_2}{C_2^2} [C_2]$ span the same ray when the equality holds. \square

Example 5.7. *Take $[C_1] = E_1$ and $[C_2] = H - E_1 - E_2$, then the intersection $\mathcal{C}_1 \cap \mathcal{C}_2$ is the ray of $H - E_2$.*

On the other hand, when $[C_1] = E_8$ and $[C_2] = 6H - 3E_1 - 2E_2 - \dots - 2E_8$, then there is no class A in $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{P}$.

Next lemma describes what could be obtained if we only use the J -inflation along two negative curves alternatively. In the following, we say a class in $\overline{\mathcal{P}_J}$ is *achieved* by formal inflations, summing and rescaling if this class could be arbitrarily approximated by these three operations starting from a class in $\overline{\mathcal{P}_J}$.

Lemma 5.8. *Suppose $h_J^+(M) = b^-(M) + 1$. Let C_1 and C_2 be two smooth J -holomorphic curves with negative intersection, and denote the boundary of \mathcal{P}_J determined by them as \mathcal{C}_1 and \mathcal{C}_2 respectively. If $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \overline{\mathcal{P}_J} \neq \emptyset$, then starting from any class $A \in \overline{\mathcal{P}_J}$, we will achieve a class in $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \overline{\mathcal{P}_J}$ by taking formal inflations along C_1 and C_2 alternatively.*

Proof. We may also assume the given class $A \in \mathcal{C}_1$, otherwise taking a maximal formal J -inflation along \mathcal{C}_1 .

We take maximal formal J -inflation along \mathcal{C}_1 and \mathcal{C}_2 alternatively. Namely, we suppose $A_0 = A$ and when $k \geq 0$,

$$\begin{aligned} A_{2k+1} &= A_{2k} + l_{2k+1}[C_2], \quad l_{2k+1} = \frac{A_{2k} \cdot [C_2]}{-C_2^2}; \\ A_{2k+2} &= A_{2k+1} + l_{2k+2}[C_1], \quad l_{2k+2} = \frac{A_{2k+1} \cdot [C_1]}{-C_1^2}. \end{aligned}$$

By calculating the coefficients l_k inductively,

$$\begin{aligned} l_1 &= \frac{A \cdot [C_2]}{-C_2^2}; \\ l_{2k+1} &= l_1 \cdot \left(\frac{([C_1] \cdot [C_2])^2}{C_1^2 \cdot C_2^2} \right)^k; \\ l_{2k} &= l_1 \cdot \frac{[C_1] \cdot [C_2]}{-C_2^2} \cdot \left(\frac{([C_1] \cdot [C_2])^2}{C_1^2 \cdot C_2^2} \right)^{k-1}. \end{aligned}$$

By Lemma 5.6, we have $([C_1] \cdot [C_2])^2 \leq C_1^2 \cdot C_2^2$.

First let us assume $([C_1] \cdot [C_2])^2 = C_1^2 \cdot C_2^2$. To consider the convergence of the classes A_k is indeed to consider the convergence of the corresponding rays of A_k . Simple calculation shows that A_k approaches the ray of $[C_2] - \frac{[C_1] \cdot [C_2]}{C_1^2}[C_1]$, which is the (unique) intersection of $\mathcal{C}_1 \cap \mathcal{C}_2$ in \mathcal{P} .

If we have $([C_1] \cdot [C_2])^2 < C_1^2 \cdot C_2^2$, we have the limit of A_k , whose value is

$$\lim_{k \rightarrow \infty} A_k = A + \frac{l_1}{1-x}([C_2] - \frac{C_1 \cdot C_2}{C_1^2}[C_1]),$$

where $x = \frac{(C_1 \cdot C_2)^2}{C_1^2 \cdot C_2^2} < 1$. When we vary the class A , we get different limiting classes. It is straightforward to check that the pairing with $[C_2]$ is

$$A \cdot [C_2] + \frac{l_1}{1-x}(C_2^2 - \frac{([C_1] \cdot [C_2])^2}{C_1^2}) = A \cdot [C_2] - \frac{A \cdot [C_2]}{1-x}(1-x) = 0.$$

Similarly the pairing with C_1 is zero as well. Since the formal inflation keeps our class in $\overline{\mathcal{P}}_J$, our conclusion follows. \square

There is a better viewpoint to see the above calculations: we are actually doing formal inflation along the ray determined by $[C'_2] = [C_2] - \frac{C_1 \cdot C_2}{C_1^2}[C_1]$. Notice $C_2'^2 < 0$ and $[C'_2] \cdot [C_1] = 0$. Hence when we do inflation along the class $[C'_2]$, the new class will keep orthogonality with C_1 . And the coefficient $\frac{l_1}{1-x}$ is nothing but the maximal inflation coefficient $\frac{A \cdot [C'_2]}{-C_2'^2}$ by simple calculation.

Lemma 5.9. *Suppose $h_J^+(M) = b^-(M) + 1$. Let C_1, C_2, \dots, C_n be smooth J -holomorphic curves with negative intersection, and denote the boundary of \mathcal{P}_J determined by them as \mathcal{C}_i . Moreover, we assume $\cap_i \mathcal{C}_i \cap \overline{\mathcal{P}}_J$ is a ray spanned by the class B . Then given any class $A \in \overline{\mathcal{P}}_J$, one could achieve*

the class B by taking formal J -inflations along C_i (as well as summing and rescaling).

Proof. If there are two C_i 's, say C_1 and C_2 , satisfy $([C_1] \cdot [C_2])^2 = C_1^2 \cdot C_2^2$, then by Lemma 5.6, $\mathbb{R}^+ B$ is the intersection $\mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{P}_J$. Thus, maximal formal J -inflations along C_1 and C_2 would approach the ray B as the limit as argued in Lemma 5.8. Hence we assume $([C_i] \cdot [C_j])^2 < C_i^2 \cdot C_j^2$ for $i \neq j$. We do induction for the n . When $n = 2$, it is Lemma 5.8.

Let us now show it for $n = 3$, whose argument suggests the general induction step. We use the viewpoint after Lemma 5.8. We first find a class in $\mathcal{C}_1 \cap \mathcal{C}_2$ by formally doing inflations for an arbitrary class $A \in \mathcal{C}_1 \cap \mathcal{P}_J$ along orthogonal classes $[C_1]$ and $[C'_2] = [C_2] - \frac{C_1 \cdot C_2}{C_1^2} [C_1]$. For the new class A_1 , we apply formal inflations along orthogonal classes $[C_1]$ and $[C'_3] = [C_3] - \frac{C_1 \cdot C_3}{C_1^2} [C_1]$. We thus obtain a class $A_2 \in \mathcal{C}_1 \cap \mathcal{C}_3$. Then we repeat this period-2 process. Notice $A_k \cdot C_1 = 0$ for all k . Hence, we are doing formal inflations along $[C'_2]$ and $[C'_3]$ alternatively. By the calculation as in Lemma 5.8, A_k converges to

$$A_1 + \frac{l'_1}{1 - x'} ([C'_3] - \frac{C'_2 \cdot C'_3}{C_2'^2} [C'_2])$$

where $l' = \frac{A_1 \cdot C'_3}{-C_3'^2}$ and $x' = \frac{(C'_2 \cdot C'_3)^2}{C_2'^2 \cdot C_3'^2}$. By the viewpoint after Lemma 5.8, we are doing formal inflations along orthogonal classes

$$[C_1], \quad [C'_2], \quad [C''_3] = [C'_3] - \frac{C'_2 \cdot C'_3}{C_2'^2} [C'_2].$$

The general induction step is a similar process of choosing orthogonal basis, an adaption of Gram-Schmidt process. Suppose in the cases of $n \leq k$, we can find orthogonal classes as positive linear combinations of original C_i and we get a class in $\cap \mathcal{C}_i$ by taking formal inflations along these orthogonal classes. Now we want to argue it for $n = k + 1$. We start with $A \in \mathcal{C}_1 \cap \mathcal{P}_J$ and obtain A_i for $i \leq k$ in turns by taking formal inflations along orthogonal classes $[C_1]$ and $[C'_{i+1}] = [C_{i+1}] - \frac{C_1 \cdot C_{i+1}}{C_1^2} [C_1]$. Repeat this period- k process. Notice all $A_k \cdot C_1 = 0$. Hence, we are doing (period- k) formal inflations along $[C'_i]$, $2 \leq i \leq k + 1$. Then the induction process implies we are actually doing formal inflations along a basis of orthogonal classes $[C''_i]$, $2 \leq i \leq k + 1$. Notice any linear combinations of $\sum a_i [C_i]$ with $a_i > 0$, especially $[C''_i]$, have non-positive square. Otherwise there is a contradiction by light cone lemma as in Lemma 5.6.

We thus finish our proof by finding the unique ray in the intersection $\cap \mathcal{C}_i$ spanned by

$$A_1 + \frac{A_1 \cdot C''_2}{-C_2''^2} [C''_2] + \cdots + \frac{A_1 \cdot C''_{k+1}}{-C_{k+1}''^2} [C''_{k+1}].$$

Notice that whatever the class A we start with, we will arrive at the same ray. \square

Example 5.10. Suppose $M = \mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$. Let the negative curves be E_3 , $E_1 - E_2$, $H - E_1 - E_2 - E_3$ and $E_2 - E_3$. Then the first three classes will determine a intersection $2H - E_1 - E_2$. Actually, we have $[C_1] = E_3$, $[C'_2] = E_1 - E_2$, $[C'_3] = [C''_3] = H - E_1 - E_2$. They are orthogonal to each other. Then our intersection ray is spanned by

$$A + (A \cdot E_3)E_3 + \frac{A \cdot (E_1 - E_2)}{2}(E_1 - E_2) + A \cdot (H - E_1 - E_2)(H - E_1 - E_2).$$

We start with different classes in the closure of \mathcal{P}_J , we will get the same ray spanned by $2H - E_1 - E_2$.

For example, if we start with H , we get $2H - E_1 - E_2$. If we start with $H - E_1$, we get

$$H - E_1 + \frac{1}{2}(E_1 - E_2) = \frac{1}{2}(2H - E_1 - E_2).$$

Now we are ready to prove Theorems 1.5 and 1.6.

Proof. (of Theorem 1.5) As there is no round boundary, each connected component is a polytope (with perhaps infinitely many faces). We denote the known almost Kähler form by ω . We want to prove that \mathcal{K}_J^c is the connected component of \mathcal{P}_J containing $[\omega]$. Since $\mathcal{K}_J^c \subset \mathcal{P}_J$ and $\mathcal{K}_J^t \cap H_J^+(M) = \mathcal{K}_J^c$, we only need to prove that \mathcal{K}_J^t contains the connected component of \mathcal{P}_J containing $[\omega]$. Since \mathcal{K}_J^t is convex, we only need to prove that each extremal ray could be achieved through the formal J -inflation, summing and rescaling.

For each extremal ray, we could find boundary hyperplanes \mathcal{C}_i 's such that each \mathcal{C}_i is determined by a smooth negative curve C_i . Thanks to Lemma 5.9, we could achieve this extremal ray by formal J -inflation along C_i 's, starting from the class $[\omega]$. After achieving each extremal ray, we could use summing and scaling to achieve the closure of the connected component of \mathcal{P}_J containing $[\omega]$ since \mathcal{K}_J^t is a convex cone.

Finally, as maximal formal J -inflation could be infinitesimally approximated by J -tamed symplectic inflation processes. We could achieve any class of the connected component of \mathcal{P}_J containing $[\omega]$ by the three operations, thus all represented by J -tamed symplectic forms. Hence \mathcal{K}_J^c is the connected component of \mathcal{P}_J containing $[\omega]$ since $\mathcal{P}_J \subset H_J^+(M)$ and $\mathcal{K}_J^t \cap H_J^+(M) = \mathcal{K}_J^c$. \square

There are many tamed almost complex manifolds whose \mathcal{P}_J has a round boundary. For example, a generic almost complex structure on manifolds with $b^+ > 1$ will do since there are not enough curves. It is also true that a generic tamed almost complex structure on $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ for $k > 9$ also has a round boundary \mathcal{P}_J . First, by Lemma 5.1, the boundary hyperplanes of \mathcal{P}_J are determined by negative curves. On the other hand, only negative curves with non-trivial Seiberg-Witten are the classes of -1 rational curves since $2e^2 = \dim_{\text{SW}}(e) + 2g_J(e) - 2 \geq -2$. Hence, generically the only negative curves are -1 rational curves. However, the condition $e \cdot E > 0$ for all $E \in \mathcal{E}_K$ does not guarantee that $e^2 \geq 0$ for $k > 9$, by taking $e = -K$ for

instance. More precisely, there is an open subset of $\partial\mathcal{P}$ whose all elements pairing positively with \mathcal{E}_K as in Remark 3.3. See the new edition of [23] for more discussions.

On the other hand, Theorem 1.6 shows in some interesting cases, the assumptions of Theorem 1.5 are satisfied. When $b^+ = 1$, there is yet another cone which is relevant to Question 1.4, the K -symplectic cone $\mathcal{C}_{M,K}$ introduced as (2).

Proof. (of Theorem 1.6) First a rational or ruled surface has $b^+ = 1$, hence $h_J^+ = b^- + 1$ holds. It is known that for rational surfaces $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with $k < 9$ or for minimal ruled surfaces, $\mathcal{C}_{M,K}$ has no round boundary. Actually, it is a polytope whose extremal rays are Cremona equivalent to H or $H - E_1$ [13]. Thus, \mathcal{P}_J has no round boundary, since it is included in and cut along by more hyperplanes from the polytope $\mathcal{C}_{M,K}$. Moreover, it is connected. When $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ and $1 < k < 9$, the boundary of \mathcal{P}_J is constituted of hyperplanes determined by curves with negative intersection because of Lemma 5.1. By Proposition 4.2, all these curves are smooth rational curves. By applying Theorem 1.5, we have $\mathcal{K}_J^c = \mathcal{P}_J$. Since $\mathcal{C}_{M,K} \subset \mathcal{P}$ in this case, we know $A_J(M)$ is a closed cone and $\mathcal{P}_J = A_J^{\vee, > 0}(M)$.

For minimal ruled surfaces (and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$), the boundary of (the closure of) the curve cone is constituted of two rays. One is the fiber class T . By a result of [22], all the J -holomorphic curves in the fiber class T are embedded. If there is a negative curve by assumption, then it is unique by Theorem 4.14. Hence the other ray is spanned by an irreducible curve class C with self-intersection $-n < 0$. Moreover, by Theorem 4.14, the class $C \cdot T = 1$. Hence for the boundary of \mathcal{P}_J , one of the ray is the fiber class T and the other is A with $A \cdot C = 0$ and $A^2 = n$.

As observed in [4], the class C is also represented by an embedded curve. We do positive inflation along T and negative inflation along C (which determines the boundary A), $B + kT + lC$ with any $k > 0$ and $0 < l < \frac{B \cdot C}{n}$ spans \mathcal{P}_J .

Hence we have constructed J -tamed symplectic form in any any class of \mathcal{P}_J . Since $b^+ = 1$, we have $\mathcal{K}_J^c = \mathcal{P}_J$. By the above discussion, the boundary of $\overline{A}_J^{\vee, > 0}$ contains two rays, one is the fiber class and the other is a class of non-negative square. Hence $\overline{A}_J^{\vee, > 0} \subset \mathcal{P}$ and thus $\mathcal{P}_J = \overline{A}_J^{\vee, > 0}$.

For the case of $M = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$, we know \mathcal{P}_J is almost polytopic in the sense that only the class $-K$ is possibly on the round boundary. In other words, any class in the interior of \mathcal{P}_J could be expressed as positive linear combinations of extremal rays. Moreover, $-K$ is the only possible curve class with non-positive self intersection which is not a rational curve. However, it does not contribute the vertices to \mathcal{P}_J . Then by Lemma 5.9, we can achieve these extremal rays by formal J -inflations along smooth rational curves. Hence we could obtain realize all the classes in \mathcal{P}_J by J -tamed symplectic

form, and thus $\mathcal{K}_J^c = \mathcal{P}_J$, as argued in Theorem 1.5. Since $\mathcal{C}_{M,K} \subset \overline{\mathcal{P}}$ in this case, we know $\mathcal{P}_J = \overline{A}_J^{\vee, >0}(M)$.

Finally, the statement $\mathcal{K}_J^c = \mathcal{K}_J^t$ follows from [18]. \square

In the case of minimal ruled surfaces, the curve cone does not necessarily be closed if the other extremal ray has square 0. However, we have irreducible positive curve classes A_n arbitrarily close to the boundary ray \mathbb{R}^+C . The author does not know how to show there is always an embedded one. If there is such one in each A_n , by Theorem 5.3, given any class B in \mathcal{P}_J , we do (positive) inflations along A_n and F to obtain $B + k_1F + k_2A_n$ which spans all \mathcal{P}_J .

We remark that actually Theorem 1.5 has more applications. One important case is when we have a smooth representation of the anti-canonical class. In this case, all the negative curves are smooth rational curves with self-intersection -1 or -2 . In fact, \mathcal{P}_J is a polytope bounded by the hyperplanes determined by those rational curves and the curve in $-K$.

On a general four dimensional symplectic manifold, we do not usually have enough embedded J -holomorphic curves, although a generic almost complex structure on manifolds with $b^+ = 1$ does have so. Thus we have Theorem 1.7.

Proof. (of Theorem 1.7) Now let us prove $\mathcal{K}_J^t = \mathcal{C}_{M,K}$ when J is in a residual set of tamed almost complex structures.

We first pick up any symplectic form ω with integral cohomology class. Let e be an integral class in $\mathcal{C}_{M,K}$. There is an integer L such that for all the integrals $l > L$, $le - [\omega]$ and $le - [\omega] - K$ are both in $\mathcal{C}_{M,K}$. By Lemma 3.4 of [16], $SW(le - [\omega])$ is nontrivial. Then $le - [\omega]$ could be represented by an embedded J -holomorphic curve for a generic J tamed by ω . We take the union of these residual subsets of ω -tame almost complex structures and denote it by $\mathcal{J}_{e,\omega}$. By Theorem 5.3, we know that $le = [\omega] + (le - [\omega])$ is represented by a J tamed symplectic form for $J \in \mathcal{J}_{e,\omega}$. Take intersection of $\mathcal{J}_{e,\omega}$ for all integral class e , we get another generic subset \mathcal{J}_ω in all ω -taming almost complex structures, since there are only countably many integral cohomology classes. Hence we have shown that for $J \in \mathcal{J}_\omega$, $\mathcal{K}_J^t = \mathcal{C}_{M,K}$.

Because the set of all integral symplectic forms is dense in the space of symplectic forms, any tamed almost complex structure is tamed by a symplectic form with integral cohomology class. Taking union of \mathcal{J}_ω for all symplectic forms with integral cohomology class, we achieve our final residual subset \mathcal{J} in all tamed almost complex structures. Hence $\mathcal{K}_J^t = \mathcal{C}_{M,K} = \mathcal{P}_J$ for a generic tamed J .

Finally, by [30] we have $\mathcal{K}_J^c \neq \emptyset$ for generic tamed J . Hence for all such J , $\mathcal{K}_J^t = \mathcal{K}_J^c$ by [18]. Thus the proof of Theorem 1.7 completes. \square

Notice the above proof is a renaissance of the argument in [16]. There is an alternative way to construct the residual set \mathcal{J} using the strategy in [30].

We endeavour to prove Question 1.4 for all tamed J rather than a residual subset. However, we may not have enough embedded J -holomorphic curves to apply the J -inflation even if we always have sufficient irreducible curves to play with the formal J -inflation.

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